DUALITY, BARYCENTRIC COORDINATES AND INTERSECTION COMPUTATION WITH GPU SUPPORT

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There are many geometric algorithms based on computation of intersection of lines, planes etc. Sometimes, very complex mathematical notations are used to express simple mathematical solutions, even if their formulation in the projective space offers much more simple solution. This paper presents solution of selected problems using principle of duality and projective space representation. It will be shown that alternative formulation in the projective space offers quite surprisingly simple solutions that lead to more robust and faster algorithms which are convenient for use within parallel architectures as GPU (Graphical Processor Units-NVIDIA) or Larrabee (Intel), which can speed up solutions of numerical problems in magnitude of 10-100.

It is shown that a solution of a system of linear equations is equivalent to generalized cross product, which leads with the duality principle to new algorithms. This is presented on a new formulation of a line in 3D given as intersection of two planes which is robust and fast, based on duality of Plücker coordinates. The presented approach can be used also for reformulation of barycentric coordinates computations on parallel architectures.

The presented approach for intersection computation is well suited especially for applications where robustness is required, e.g. large GIS/CAD/CAM systems etc.

Keywords: computer graphics; homogeneous coordinates; Plücker coordinates; principle of duality; line and plane intersections computation; projective geometry; barycentric coordinates

Notation used:
E^n - n-dimensional Euclidean space,
D^n - n-dimensional Dual space,
P^n - n-dimensional Projective space of E^n or D^n,
$\boldsymbol{X}$ - vector in Euclidean or Dual spaces,
$\boldsymbol{x}$ - vector in Projective space,
$\boldsymbol{x}_k^{(i)}$ - value of the i-th coordinate of the vector $\boldsymbol{x}_k$,
i.e. $\boldsymbol{x}_k^{(2)} = y_k$ etc.
$\boldsymbol{a} \times \boldsymbol{b}$ - cross-product of $\boldsymbol{a}$, $\boldsymbol{b}$ vectors,
$\boldsymbol{a} \,.\, \boldsymbol{b}$ or $\boldsymbol{a}^T \boldsymbol{b}$ - dot-product of $\boldsymbol{a}$, $\boldsymbol{b}$ vectors.

1. Introduction

The homogeneous coordinates are used in computer graphics and related fields to represent geometric transformations, projections. They are often thought to be just a mathematical tool to enable representation of fundamental geometric transformations by matrix or vector multiplications. The homogeneous coordinates are not the only ones available. Nevertheless, they are often used for point/line/plane description in the projective space.

Algorithms for intersection computations are usually performed in the Euclidean space using the Euclidean coordinates. Primitives are converted from the homogeneous coordinates to the Euclidean coordinates before computation and the result is converted back to the homogeneous coordinates for future manipulations. This is quite visible especially within the graphical pipeline.

Unfortunately, these conversions are time consuming and require division operations, which cause instability and decrease robustness in some situations.

There are some simple problems like intersection of lines or planes, where computer scientists have trouble if the Euclidean coordinates are used. On the other hand, the homogeneous coordinates cause some difficulties in development of new algorithms. Of course, it is necessary to understand principle of the projective geometry and geometrical interpretation of the projective space.

2. Projective geometry

The homogeneous coordinates are mostly introduced with geometric transformations concepts and used for the projective space representation. Many books and papers define mathematically how to make transformations from the homogeneous coordinates to the Euclidean coordinates and vice versa. Nevertheless, geometrical interpretation is missing in nearly all publications. Therefore, the question is how to imagine the projective space P^2 and representations of elements.

Conversion from the homogeneous coordinates to the Euclidean coordinates is defined for E^2 case as:

$$X = x / w \quad Y = y / w \tag{1}$$

where: $w \neq 0$, point $\boldsymbol{x}$ = [x, y, w]^T and $\boldsymbol{x} \in P^2$,
$\boldsymbol{X}$ = [X, Y]^T and $\boldsymbol{X} \in E^2$,
if $w = 0$ then $\boldsymbol{x}$ represents "an ideal point", that is a point in infinity.

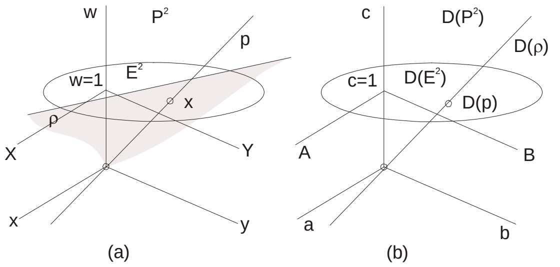


Figure 1:
Euclidean, projective and dual space representations

Let us consider a situation at Fig.1.a. We can see that the point $\boldsymbol{X} \in E^2$ in the Euclidean space is actually a line p in the projective space P^2 passing the given point $\boldsymbol{X} \in E^2$ at the plane $w = 1$ (that is the Euclidean space actually) and the origin of the projective space P^2. It means that all the points $\boldsymbol{x} \in P^2$ of the line (excluding [$0, 0, 0$]^T) represent the same point in the Euclidean space. Similarly, transformation for the E^3 case is defined as:

$$X = x / w \quad Y = y / w \quad Z = z / w \tag{2}$$

where: $w \neq 0$, point $\boldsymbol{x}$ = [x, y, z, w]^T and $\boldsymbol{x} \in P^3$,
$\boldsymbol{X}$ = [X, Y, Z]^T and $\boldsymbol{X} \in E^3$.

Let us assume the Euclidean space E^2, see Fig.1.a. We actually use the projective space whenever we use the implicit representation for graphical elements.
Let us imagine that the Euclidean space E^2 is represented as a plane $w = 1$. For simplicity, let us consider a line p defined as:

$$aX + bY + c = 0 \tag{3}$$

We can multiply it by $w \neq 0$ and we get:

$$ax + by + cw = 0 \tag{4}$$

It is actually a plane in the projective space P^2 (excluding the point [$0, 0, 0$]^T) passing through the origin. The vector of coefficients $\boldsymbol{p}$ represents the line $p \in E^2$:

$$\boldsymbol{p} = [a, b, c]^T \tag{5}$$

Let us assume a dual representation, see Fig.1.b. In the dual representation in which the point [a, b, c]^T actually represents a line $D(p) \in D(E^2)$ given by the point [a, b, c]^T and the origin of the dual space, see [1], [2] for details on projective geometry.
It is necessary to note that any $\xi \neq 0$ can multiply the Eq.4 without any effect to the geometry. It means that there will be different vectors of coefficients $\boldsymbol{p}$ that will represent the same line $p \in E^2$.
In the dual coordinate system, those points will form a line $D(p)$. We can project the line $D(p)$ e.g. to a plane with $c = 1$ and we get a point. The line $p \in E^2$ is actually represented in projective space by a plane $\rho \in P^2$ (the origin [$0,0,0$]^T is excluded). It means that the line $p \in E^2$ is a point in the **dual representation** $D(p) \in D(E^2)$ and vice versa.
On the other hand, there is a phenomenon of a principle of duality that can be used for derivation of some useful formula.

3. Principle of duality

The principle of duality in E^2 states that any theorem remains true when we interchange the words "point" and "line", "lie on" and "pass through", "join" and "intersection", "collinear" and "concurrent" and so on. Once the theorem has been established, the dual theorem is obtained as described above, see [3], [4] for details.
In other words, the principle of duality says that in all theorems it is possible to substitute the term "point" by the term "line" and the term "line" by the term "point" etc. in E^2 and the given theorem stays valid. Similar duality is valid for E^3 as well, i.e. the terms "point" and "plane" are dual etc. This helps a lot to solve some geometrical problems.

3.1. E^2 case

In the E^2 case, parameters of a line given by two points or an intersection point of two lines are computed very often. We will use the duality principle in which a point is dual to a line and vice versa.
In the first case, the solution is simple if the points are not in the homogeneous coordinates. If they are given in the homogeneous coordinates, the coordinates are converted to the Euclidean coordinates and then parameters of the line are computed.
In the second case, a linear system of equations of the degree two is usually solved and division is to be performed. It is necessary to note that any division operation decreases robustness of computation.
A new approach performing an appropriate computation in projective space will be presented. It will allow us to avoid division operations.

Definition$_1$
The cross-product of two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in E^2$, if given in the homogeneous coordinates, is defined as (if $w = 1$ the standard formula is obtained):

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \tag{6}$$

where: $\boldsymbol{i} = [1,0,0]^T$, $\boldsymbol{j} = [0,1,0]^T$, $\boldsymbol{k} = [0,0,1]^T$

$$w_1 w_2 (\mathbf{X}_1 \times \mathbf{X}_2) = \mathbf{x}_1 \times \mathbf{x}_2 =$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{vmatrix} = w_1 w_2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{x_1}{w_1} & \frac{y_1}{w_1} & 1 \\ \frac{x_2}{w_2} & \frac{y_2}{w_2} & 1 \end{vmatrix} = \tag{7}$$
$$w_1 w_2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{vmatrix}$$

Theorem$_1$
Let two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in E^2$ be given in the projective space. Then a line $p \in E^2$ defined by those two points is determined as a cross-product:

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \tag{8}$$

where: $\boldsymbol{p} = [a, b, c]^T$

Proof$_1$
Let the line $p \in E^2$ is defined as:

$$ax + by + c = 0 \tag{9}$$

The end-points must satisfy Eq.9 and therefore $\boldsymbol{x}_1^T \boldsymbol{p} = 0$ and $\boldsymbol{x}_2^T \boldsymbol{p} = 0$, i.e.

$$\begin{bmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

It can be seen that it is a standard formula [5] if the Eq.7 is used:

$$a = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} \quad b = -\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} \quad c = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \tag{11}$$

and therefore the cross-product defines the line p, i.e.

$$\boldsymbol{p} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \tag{12}$$

□

Note: It can be seen that Eq.8 is valid also for cases when $w \neq 0$ and $w \neq 1$. Coefficients a, b, c can be determined as sub-determinants in the Eq.10. The proof is left to a reader.

Now we can apply the principle of duality directly.

Theorem$_2$
Let two lines $\boldsymbol{p}_1, \boldsymbol{p}_2 \in E^2$ be given in the projective space. Then a point $\boldsymbol{x}$ defined as an intersection of those two lines is determined as a cross product:

$$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2. \tag{13}$$

where: $\boldsymbol{x} = [x, y, w]^T$

Proof$_2$
This is a direct consequence of the principle of duality application.

$$\boldsymbol{x} = \boldsymbol{p}_1 \times \boldsymbol{p}_2 = \begin{vmatrix} x & y & w \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \tag{14}$$

where: $\boldsymbol{x} = [x, y, w]^T$

□

These two theorems are very important as they enable us to handle some problems defined in the homogeneous coordinates directly and make computations quite effective.
Direct impact of these two theorems is that it is very easy to compute a line given by two points in E^2 and an intersection point of two lines in E^2 as well. The presented approach is convenient if vector-vector operations are supported, especially for GPU applications. Note that we do not need to solve linear system of equations to find the intersection point of two lines and if the result can remain in the homogeneous coordinates, no division operation is needed.
Of course, there is a question, how to handle the E^3 cases.

3.2. *E^3 case*

The E^3 case is a little bit complicated as the projective geometry and duality offer more possibilities, but generally a point is dual to a plane and vice versa. So let us explore how to find:

- a plane defined by three points given in the homogeneous coordinates,
- an intersection point of three planes.

To find a plane is simple if points are converted to the Euclidean coordinates. It requires use of the division operation and therefore robustness is decreased in general.
Let us explore the extension possibility of the E^2 cases, as discussed above, to the E^3 case.

Definition$_2$
The cross-product of three vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ is defined as:

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix} \quad (15)$$

where: $\boldsymbol{i} = [1,0,0,0]^T$, $\boldsymbol{j} = [0,1,0,0]^T$, $\boldsymbol{k} = [0,0,1,0]^T$, $\boldsymbol{l} = [0,0,0,1]^T$

Theorem$_3$
Let three points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ be given in the projective space. Then a plane $\boldsymbol{\rho} \in E^3$ defined by those three points is determined as:

$$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 \quad (16)$$

Proof$_3$
Let the plane $\boldsymbol{\rho} \in E^3$ be defined as:

$$ax + by + cz + d = 0 \quad (17)$$

It can be seen that:

$$a = \begin{vmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{vmatrix} \qquad b = -\begin{vmatrix} x_1 & z_1 & w_1 \\ x_2 & z_2 & w_2 \\ x_3 & z_3 & w_3 \end{vmatrix}$$

$$c = \begin{vmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \\ x_3 & y_3 & w_3 \end{vmatrix} \qquad d = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad (18)$$

that is the cross-product that defines a plane ρ if three points are given and therefore:

$$\boldsymbol{\rho} = \boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 \quad (19)$$

□

Note: It can be seen that it is a standard formula for the case $w = 1$ [5]. The proof is left to a reader.

As a point is dual to a plane, a plane is dual to a point we can use the principle of duality directly, now.

Theorem$_4$
Let three planes $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ be given in the projective space. Then a point $\boldsymbol{x}$, which is defined as the intersection point of those three planes, is determined as:

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 \quad (20)$$

where: $\boldsymbol{x} = [\,x, y, z, w]^T$

Proof$_4$
This is a direct consequence of the principle of duality application:

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2 \times \boldsymbol{\rho}_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \quad (21)$$

where: $\boldsymbol{x} = [\,x, y, z, w]^T$
□

These two theorems are very important as they enable us to handle some problems defined in the homogeneous coordinates efficiently and make computations quite effective. Even more, if an input is in the Euclidean or homogeneous coordinates and output can be in the homogeneous coordinates, no division is needed. It means that we have robust computation of an intersection point.
Direct impact of these two theorems is that it is very easy to compute a plane in the E^3 given by three points in the E^3 and compute an intersection point determined as an intersection of three planes in the E^3. Of course, there is a question, how to handle lines in the E^3 or P^3 cases.
The above mentioned formulae Eq.16 and Eg.20 are not known in general and the authors present explicit formulae for the Euclidean coordinates, i.e. for $w = 1$, see Eq.41 and formula 44.

3.3. *Line in E^3 defined parametrically*

Let us consider a little bit more difficult problems formulated as follows:

1. determine a line $q \in E^3$ if given by two points $\boldsymbol{x}_i$,
2. determine a line $q \in E^3$ if given by two planes $\boldsymbol{\rho}_i$.

if the parametric form is required.

These problem formulations seem to be trivial problems if $w_i = 1$ and the division operation are permitted.

On the other hand, a classic rule for robustness is to "postpone division operation to the last moment possible". Even if division is permitted, the 2nd case seems to be more difficult not only from the robustness point of view as the line is considered as an intersection of two planes, i.e. a common solution of their implicit equations.

We will derive a new method for determination of a line in the E^3 for those two possible cases without use of division directly in the projective space.

The Plücker coordinates will be used as they can help us to formalize and resolve this problem efficiently.

4. Plücker coordinates

The formulae presented above enable us to handle points and planes in E^3. Nevertheless, it is necessary to have a way to handle lines in the E^3 in the parametric form using the homogeneous coordinates as well and avoid the division operations, too. A parametric form for a line given by two points in the Euclidean coordinates is given as:

$$\boldsymbol{X}(t) = \boldsymbol{X}_1 + (\boldsymbol{X}_2 - \boldsymbol{X}_1)\, t \tag{22}$$

where: t is a parameter $t \in (-\infty, \infty)$.

This is straightforward for the Euclidean coordinates and for the homogeneous coordinates if the division operation is permitted. It is necessary to represent a position and a direction, see Eq.22. The question is how to make it directly in the projective space using the homogeneous coordinates. Therefore, the Plücker coordinates will be introduced to resolve the situation. Another approach using the Grassmann coordinate system can be found in [6].

Let us consider two points in the homogeneous coordinates:

$$\boldsymbol{x}_1 = [x_1, y_1, z_1, w_1]^{\mathrm{T}} \qquad \boldsymbol{x}_2 = [x_2, y_2, z_2, w_2]^{\mathrm{T}} \tag{23}$$

The Plücker coordinates l_{ij} are defined as follows:

$$\begin{aligned} l_{41} &= w_1x_2 - w_2x_1 & l_{23} &= y_1z_2 - y_2z_1 \\ l_{42} &= w_1y_2 - w_2y_1 & l_{31} &= z_1x_2 - z_2x_1 \\ l_{43} &= w_1z_2 - w_2z_1 & l_{12} &= x_1y_2 - x_2y_1 \end{aligned} \tag{24}$$

It is possible to express the Plücker coordinates as

$$l_{ij} = \boldsymbol{x}_1^{(i)}\boldsymbol{x}_2^{(j)} - \boldsymbol{x}_2^{(i)}\boldsymbol{x}_1^{(j)} \tag{25}$$

alternatively, as an anti-symmetric matrix $\boldsymbol{L}$:

$$\boldsymbol{L} = \boldsymbol{x}_1\boldsymbol{x}_2^T - \boldsymbol{x}_2\boldsymbol{x}_1^T \tag{26}$$

where: $l_{ij} = -\, l_{ji}$ and $l_{ii} = 0$.

Let us define two vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ as:

$$\boldsymbol{\omega} = [l_{41}, l_{42}, l_{43}]^{\mathrm{T}} \qquad \boldsymbol{v} = [l_{23}, l_{31}, l_{12}]^{\mathrm{T}} \tag{27}$$

It means that $\boldsymbol{\omega}$ represents the "directional vector", while $\mathbf{v}$ represents the "positional vector". It can be seen that for the Euclidean space ($w = 1$) we get:

$$\boldsymbol{X}_2 - \boldsymbol{X}_1 = \boldsymbol{\omega} \qquad \boldsymbol{X}_1 \times \boldsymbol{X}_2 = \boldsymbol{v} \tag{28}$$

where: $\boldsymbol{X}_{\mathrm{i}} = [x_i, y_i, z_i]^{\mathrm{T}}/w_i$ are points in the Euclidean coordinates.

For general case $w_i \neq 1$ when $\boldsymbol{x}_i$ are not ideal points, i.e. $w_i \neq 0$ we get:

$$\boldsymbol{X}_2 - \boldsymbol{X}_1 = \left(\frac{x_2}{w_2} - \frac{x_1}{w_1}, \frac{y_2}{w_2} - \frac{y_1}{w_1}, \frac{z_2}{w_2} - \frac{z_1}{w_1}\right) \tag{29}$$

It can be seen that for the projective space, vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}$ can be expressed as:

$$\begin{gathered} \boldsymbol{\omega} = w_2 w_1 (\mathbf{X}_2 - \mathbf{X}_1) = \\ (x_2 w_1 - x_1 w_2\,,\, y_2 w_1 - y_1 w_2\,,\, z_2 w_1 - z_1 w_2) \\ = (l_{41}, l_{42}, l_{43}) \end{gathered} \tag{30}$$

and

$$\begin{gathered} \boldsymbol{v} = w_2 w_1 (\boldsymbol{X}_1 \times \boldsymbol{X}_2) = \\ (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1) = \\ (l_{23}, l_{31}, l_{12}) \end{gathered} \tag{31}$$

The Eq.30 and Eq.31 show the relation between vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}$ and the Plücker coordinates l_{ij}. In 1871 Klein derived that $\boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{v} = 0$ [7], i.e. in the Plücker coordinates:

$$l_{23} * l_{41} + l_{31} * l_{42} + l_{12} * l_{43} = 0 \tag{32}$$

This is a homogeneous equation of degree 2 and therefore the solution lies on a 4-dimensional quadratic hyper-surface [8]. If q is a point on a line $\boldsymbol{q}(t) = \boldsymbol{q}_1 + \boldsymbol{\omega}\, t$ given by the Plücker coordinates, it must satisfy equation:

$$\boldsymbol{\omega} \times \boldsymbol{q} = \boldsymbol{v} \tag{33}$$

Let $\boldsymbol{X}_2 - \boldsymbol{X}_1 = \boldsymbol{\omega}$ and $\boldsymbol{X}_1 \times \boldsymbol{X}_2 = \boldsymbol{v}$. A point on the line $\boldsymbol{q}(t) = \boldsymbol{q}_1 + \boldsymbol{\omega}\, t$ is defined as:

$$q(t) = \frac{v \times \omega}{\|\omega\|^2} + \omega t \tag{34}$$

Please, see Appendix B for derivation of this formula. It should be noted that for $t = 0$ we do not get the point X_1. If $\|\omega\| = 0$ the given points are equal.

The Eq.34 defines a line $q(t)$ in the E^3 by two points x_1 and x_2 given in the homogeneous coordinates. Of course, we can avoid the division operation easily using homogeneous notation for a scalar value $\hat{q}(t)$, as follows:

$$\hat{q}(t) = \begin{bmatrix} v \times \omega + t\omega\|\omega\|^2 \\ \|\omega\|^2 \end{bmatrix} \tag{35}$$

and the resulting line is defined directly in the projective space P^3.

Let us imagine that we have to solve the second problem, i.e. a line defined as an intersection of two given planes ρ_1 and ρ_2 in the Euclidean space:

$$\rho_1 = [a_1, b_1, c_1, d_1]^T \quad \rho_2 = [a_2, b_2, c_2, d_2]^T \tag{36}$$

It is well known that the directional vector s of the line is given by those two planes as a ratio:

$$s_x : s_y : s_z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{37}$$

that is actually the ratio $l_{23} : l_{31} : l_{12}$ if the principle of duality is used, i.e. vector of $[a_i, b_i, c_i, d_i]^T$ instead of $[x_i, y_i, z_i, w_i]^T$ is used, and it defines the vector v instead of ω.
Now we can apply the principle of duality as we can interchange the terms "point" and "plane" and exchange v and ω in the Eq.34 and we get:

$$q(t) = \frac{\omega \times v}{\|v\|^2} + vt \tag{38}$$

and similarly to the Eq.35, the formula for the line in the homogeneous coordinates is given as:

$$\hat{q}(t) = \begin{bmatrix} \omega \times v + tv\|v\|^2 \\ \|v\|^2 \end{bmatrix} \tag{39}$$

If $\|v\| = 0$ then the given planes are parallel.

It means that we have obtained the known formula for an intersection of two planes ρ_1, ρ_2 in the Euclidean coordinates, see [5]:

$$q(t) = q_0 + n_3 t \tag{40}$$

where: $n_3 = n_1 \times n_2$, $q_0 = [X_0, Y_0, Z_0]^T$ and planes
$\rho_1 : n_1^T x + d_1 = 0 \quad \rho_2 : n_2^T x + d_2 = 0$

The intersection point X_0 of three planes in the Euclidean coordinates is defined as:

$$X_0 = \frac{d_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} - d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{DET}$$

$$Y_0 = \frac{d_2 \begin{vmatrix} a_3 & c_3 \\ a_1 & c_1 \end{vmatrix} - d_1 \begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix}}{DET}$$

$$DET = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If a line is defined by two points and $\|\omega\| = 1$, i.e. the directional vector is normalized, we get Eq.34 and the line is simply determined as:

$$q(t) = v \times \omega + \omega t \tag{42}$$

If a line is defined by two planes and $\|v\| = 1$, i.e. the positional vector is normalized, we get Eq.38 and the line is simply determined as:

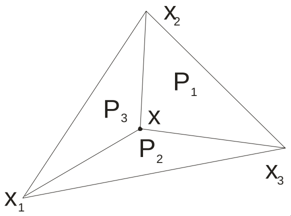
$$q(t) = \omega \times v + vt \tag{43}$$

Those formulae are well known if the Euclidean coordinates are used.

Note:
It is possible to define vectors v and ω for the plane intersection case as $v = [l_{41}, l_{42}, l_{43}]^T$ and $\omega = [l_{23}, l_{31}, l_{12}]^T$, i.e. with swapped Plücker vectors, and have the same equation for the line $q(t)$ but the symbols would have different interpretation – that is the reason, why the priority was given to different notation for those two cases.

5. Barycentric coordinates

In computer graphics Euclidean or homogeneous coordinates are widely used as well as parametric formulations, e.g. triangles, parametric patches etc. The barycentric coordinates have many useful and interesting properties, see [3], [I.1] for details.



Barycentric coordinates in E^2
Figure 2

Let us consider a triangle with vertices $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, see Fig.2. The position of any point $\mathbf{X} \in E^2$ can be expressed as

$$\begin{aligned} a_1 X_1 + a_2 X_2 + a_3 X_3 &= X \\ a_1 Y_1 + a_2 Y_2 + a_3 Y_3 &= Y \end{aligned} \qquad (44)$$

if we add an additional condition

$$a_1 + a_2 + a_3 = 1 \qquad (45)$$

we get a system of linear equations. The coefficients a_i are called barycentric coordinates of the point **X**. The point **X** is inside the triangle **if and only if** $0 \le a_i \le 1$, i = 1,…,3. It is useful to know that

$$a_i = \frac{P_i}{P} \quad \text{i} = 1,...,3 \qquad (46)$$

where: P is the area of the given triangle and P_i is the area of the i-th subtriangle.

Note: The barycentric coordinates can easily be converted into the usual parametric form. It can be seen that $a_1 = 1 - a_2 - a_3$. Substituting into eq.44 we obtain

$$(1 - a_2 - a_3) X_1 + a_2 X_2 + a_3 X_3 = X \qquad (47)$$

i.e.

$$X_1 + a_2 (X_2 - X_1) + a_3 (X_3 - X_1) = X \qquad (48)$$

and finally we get

$$a_2 (X_2 - X_1) + a_3 (X_3 - X_1) = X - X_1 \qquad (49)$$

It is the standard formula usually used. Similarly, it may be used for other coordinates.

It can be seen that a system of linear equations, eqs.44-45, must be solved, i.e.

$$\mathbf{A}\boldsymbol{\alpha} = \boldsymbol{\beta} \qquad (50)$$

where: $\boldsymbol{\alpha} = [a_1, a_2, a_3]^T$, $\boldsymbol{\beta} = [X, Y, 1]^T$ and

$$\mathbf{A} = \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix}$$

and division operations must be used to solve this linear system of equations. In some cases, especially when the triangles are very thin, there might be a severe problem with the stability of the solution. The non-homogeneous system of linear equations $\mathbf{A}\boldsymbol{\alpha} = \boldsymbol{\beta}$ can be transformed into a homogeneous linear system

$$\begin{aligned} b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X &= 0 \\ b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + b_4 Y &= 0 \\ b_1 + b_2 + b_3 + b_4 &= 0 \end{aligned} \qquad (51)$$

where: $b_4 \ne 0$ and $b_i = -a_i b_4 \quad i = 1,...,3$.

Rewriting this system in a matrix form, we get

$$\begin{bmatrix} X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0} \qquad (52)$$

or in the matrix form

$$\mathbf{B}\,\mathbf{b} = \mathbf{0} \quad or \quad [\mathbf{A} \mid \mathbf{X}][\mathbf{b}] = \mathbf{0} \qquad (53)$$

where: $\mathbf{b} = [b_1, b_2, b_3, b_4]^T$, $\mathbf{X} = [X, Y, 1]^T$,

$$\mathbf{A} = \begin{bmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \\ 1 & 1 & 1 \end{bmatrix}$$ and $\mathbf{B} = [\mathbf{A} \mid \mathbf{X}]$

In another way, we are looking for a vector $\boldsymbol{\tau}$, see eq.4, that satisfies the condition

$$\boldsymbol{\tau}^T \mathbf{b} = 0 \qquad (54)$$

where: $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4]^T$

This equation can be expressed using the determinant form as:

$$\det \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ X_1 & X_2 & X_3 & X \\ Y_1 & Y_2 & Y_3 & Y \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 \qquad (55)$$

It is obvious that it can be formally written as:

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \mathbf{w} \qquad (56)$$

where: $\mathbf{b} = [b_1, b_2, b_3, b_4]^T \quad \boldsymbol{\xi} = [X_1, X_2, X_3, X]^T$

$\boldsymbol{\eta} = [Y_1, Y_2, Y_3, Y]^T \quad \mathbf{w} = [1,1,1,1]^T$

and the barycentric coordinates of the point $\mathbf{X}$ are given as $a_1 = -\frac{b_1}{b_4}$, $a_2 = -\frac{b_2}{b_4}$, $a_3 = -\frac{b_3}{b_4}$

We can use the Plücker coordinates notation and write

$a_i = (-b_i : b_4) \quad i = 1,...,3$.

If $b_4 = 0$, the triangle is degenerated to a line segment or to a point, i.e. it is a singular case, which can be correctly detected.

The given point $\mathbf{X}$ is inside the given **triangle if and only if** $0 \le a_i \le 1$, i = 1,…,3. This condition is a little bit more complicated for the homogeneous representation and can be expressed by a sequence

if $b_4 > 0$ **then** $0 \le -b_i \le b_4$

else $b_4 \le -b_i \le 0 \quad i = 1,...,3$

This is a very important result as it means that **we do not need the division operation** for testing whether the given point $\mathbf{X}$ is inside the given triangle!

In many applications, the vertices of the given triangle and the given point $\mathbf{X}$ can be given in homogeneous coordinates. Let us explore how the barycentric coordinates could be computed in this case.

The linear system of equations for the barycentric coordinates can be rewritten as:

$$
\begin{aligned}
& a_1 \frac{x_1}{w_1} + a_2 \frac{x_2}{w_2} + a_3 \frac{x_3}{w_3} = \frac{x}{w} \\
& a_1 \frac{y_1}{w_1} + a_2 \frac{y_2}{w_2} + a_3 \frac{y_3}{w_3} = \frac{y}{w} \qquad (57) \\
& a_1 + a_2 + a_3 = 1
\end{aligned}
$$

where: $\mathbf{x}_i = [x_i, y_i, w_i]^T$ represents the i-th vertex triangle in the homogeneous coordinates and $\mathbf{x} = [x, y, w]^T$ is the given point in the homogeneous coordinates.

We can multiply the linear system by $w \neq 0$, $w_i \neq 0 \quad i = 1,...,3$ and substitute:

$$
\begin{aligned}
& b_1 = -a_1 w_2 w_3 w \qquad b_2 = -a_2 w_1 w_3 w \\
& b_3 = -a_3 w_1 w_2 w \qquad b_4 = w_1 w_2 w_3 w
\end{aligned} \qquad (58)
$$

Thus we get:

$$
\begin{aligned}
& b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x = 0 \\
& b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y = 0 \qquad (59) \\
& b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 w = 0
\end{aligned}
$$

and in the matrix notation:

$$
\begin{bmatrix} x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ w_1 & w_2 & w_3 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{0} \qquad (60)
$$

We are looking for a vector $\boldsymbol{\tau}$ that satisfies the following equation:

$$
\boldsymbol{\tau}^T \mathbf{b} = 0 \qquad (61)
$$

where: the vector $\boldsymbol{\tau}$ is defined as $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4]^T$

Then the solution is defined as

$$
\det \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ w_1 & w_2 & w_3 & w \end{vmatrix} = 0 \qquad (62)
$$

and we can formally write

$$
\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \mathbf{w} \qquad (63)
$$

where: $\mathbf{b} = [b_1, b_2, b_3, b_4]^T \quad \boldsymbol{\xi} = [x_1, x_2, x_3, x]^T$

$\boldsymbol{\eta} = [y_1, y_2, y_3, y]^T \quad \mathbf{w} = [w_1, w_2, w_3, w]^T$

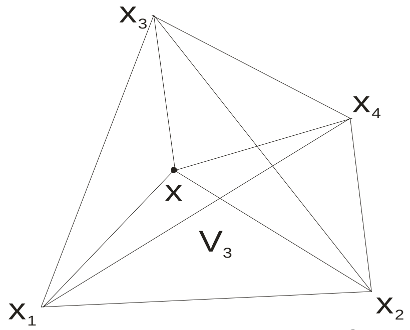
Of course, the conditions in the case that the point is inside the given triangle are slightly more complex, and the condition $0 \le a_i \le 1 \;\; i = 1,...,3$ can be expressed by the following criteria:

$$
\begin{aligned}
& 0 \le (-b_1 : w_2 w_3 w) \le 1 \\
& 0 \le (-b_2 : w_1 w_3 w) \le 1 \qquad (64) \\
& 0 \le (-b_3 : w_1 w_2 w) \le 1
\end{aligned}
$$

This means that the barycentric coordinates can be computed **without using the division operation** even if the vertices of the given triangle and the point **x** are given in homogeneous coordinates. Therefore the approach presented here is more robust than the direct computation, i.e. normalizing the vertices and point coordinates into Euclidean coordinates and standard barycentric coordinates computation. In addition, the test if a point is inside the given triangle is consequently more robust.

Of course, there is a natural question: is it possible to extend the above mentioned approach to the E^3 case?

Let us consider the E^3 case, where the “point in a tetrahedron” test is similar to the “point in a triangle” test in E^2, see Fig.3.



Barycentric coordinates in E^3

Figure 3

It can be seen that the barycentric coordinates are given as

$$
\begin{aligned}
& a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 = X \\
& a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 = Y \\
& a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4 = Z \qquad (65) \\
& a_1 + a_2 + a_3 + a_4 = 1
\end{aligned}
$$

It is useful to know that

$$a_i = \frac{V_i}{V} \quad \text{i = 1,...,3} \tag{66}$$

where: V is the volume of the given tetrahedron and V_i is the volume of the i-th sub-tetrahedron.

The non-homogeneous system of linear equations can be transformed into a homogeneous linear system of equations

$$\begin{aligned} &b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 + b_5X = 0 \\ &b_1Y_1 + b_2Y_2 + b_3Y_3 + b_4Y_4 + b_5Y = 0 \\ &b_1Z_1 + b_2Z_2 + b_3Z_3 + b_4Z_4 + b_5Z = 0 \\ &b_1 + b_2 + b_3 + b_4 + b_5 = 0 \end{aligned} \tag{67}$$

where: $b_5 \neq 0$ and $b_i = -a_i b_5 \quad i = 1,...,4$

Rewriting this system in matrix form, we get

$$\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X \\ Y_1 & Y_2 & Y_3 & Y_4 & Y \\ Z_1 & Z_2 & Z_3 & Z_4 & Z \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \mathbf{0} \tag{68}$$

i.e.

$$\mathbf{B}\,\mathbf{b} = \mathbf{0} \quad or \quad [\mathbf{A} \mid \mathbf{X}][\mathbf{b}] = \mathbf{0} \tag{69}$$

where: $\mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^T$, $\mathbf{X} = [X, Y, Z, 1]^T$,

$$\mathbf{A} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ Z_1 & Z_2 & Z_3 & Z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{B} = [\mathbf{A} \mid \mathbf{X}]$$

Again, we are looking for a vector $\boldsymbol{\tau}$ that satisfies the equation

$$\boldsymbol{\tau}^T \mathbf{b} = 0 \tag{70}$$

where: $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3, \tau_4, \tau_5]^T$

The equation can be expressed using a determinant form as:

$$\det \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \\ X_1 & X_2 & X_3 & X_4 & X \\ Y_1 & Y_2 & Y_3 & Y_4 & Y \\ Z_1 & Z_2 & Z_3 & Z_4 & Z \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0 \tag{71}$$

It can be seen that we can formally write again:

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \boldsymbol{\zeta} \times \mathbf{w} \tag{72}$$

where: $\mathbf{b} = [b_1, b_2, b_3, b_4, b_5]^T$

$\boldsymbol{\xi} = [X_1, X_2, X_3, X_4, X]^T$ $\boldsymbol{\eta} = [Y_1, Y_2, Y_3, Y_4, Y]^T$

$\boldsymbol{\zeta} = [Z_1, Z_2, Z_3, Z_4, Z]^T$ $\mathbf{w} = [1, 1, 1, 1, 1]^T$

This means that the barycentric coordinates of the point $\mathbf{X}$ are given as:

$$a_1 = -\frac{b_1}{b_5}, \quad a_2 = -\frac{b_2}{b_5}, \quad a_3 = -\frac{b_3}{b_5}, \quad a_4 = -\frac{b_4}{b_5} \tag{73}$$

or if we use the Plücker coordinates notation, they are given as

$a_i = (-b_i : b_5) \quad i = 1,...,4$.

The given point $\mathbf{X}$ is inside the given tetrahedron **if and only if** $0 \le a_i \le 1$, i = 1,…,4.

This condition can be expressed by the following sequence

if $b_5 > 0$ **then** $0 \le -b_i \le b_5$

else $b_5 \le -b_i \le 0$

If $b_5 = 0$, the tetrahedron is degenerated to a triangle or to a line segment or to a point, i.e. singular cases that can be correctly detected.

Let us again consider a case when the tetrahedron vertices and the given point are in homogeneous coordinates.

The linear system of equations can be rewritten as:

$$\begin{aligned} &a_1\frac{x_1}{w_1} + a_2\frac{x_2}{w_2} + a_3\frac{x_3}{w_3} + a_4\frac{x_4}{w_4} = \frac{x}{w} \\ &a_1\frac{y_1}{w_1} + a_2\frac{y_2}{w_2} + a_3\frac{y_3}{w_3} + a_4\frac{y_4}{w_4} = \frac{y}{w} \\ &a_1\frac{z_1}{w_1} + a_2\frac{z_2}{w_2} + a_3\frac{z_3}{w_3} + a_4\frac{z_4}{w_4} = \frac{z}{w} \\ &a_1 + a_2 + a_3 + a_4 = 1 \end{aligned} \tag{74}$$

where: $\mathbf{x}_i = [x_i, y_i, z_i, w_i]^T$ represents the i-th vertex coordinates in the homogeneous coordinates.

We can multiply the linear system of equations by $w \neq 0$, $w_i \neq 0 \quad i = 1,...,4$ and substitute

$$\begin{aligned} &b_1 = -a_1 w_2 w_3 w_4 w \qquad b_2 = -a_2 w_1 w_3 w_4 w \\ &b_3 = -a_3 w_1 w_2 w_4 w \qquad b_4 = -a_4 w_1 w_2 w_3 w \\ &b_5 = w_1 w_2 w_3 w_4 \end{aligned} \tag{75}$$

This results into a standard homogeneous linear system:

$$\begin{aligned} &b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x = 0 \\ &b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4 + b_5y = 0 \\ &b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4 + b_5z = 0 \\ &w_1b_1 + w_2b_2 + w_3b_3 + w_4b_4 + w\, b_5 = 0 \end{aligned} \tag{76}$$

that can be expressed in the matrix form as:

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x \\ y_1 & y_2 & y_3 & y_4 & y \\ z_1 & z_2 & z_3 & z_4 & z \\ w_1 & w_2 & w_3 & w_4 & w \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \mathbf{0} \qquad (77)$$

We are looking for a vector $\boldsymbol{\tau}$ that satisfies the equation

$$\boldsymbol{\tau}^T \mathbf{b} = 0 \qquad (78)$$

where: the vector $\boldsymbol{\tau} = \left[\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\right]^T$ is defined as

$$\det \begin{vmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \\ x_1 & x_2 & x_3 & x_4 & x \\ y_1 & y_2 & y_3 & y_4 & y \\ z_1 & z_2 & z_3 & z_4 & z \\ w_1 & w_2 & w_3 & w_4 & w \end{vmatrix} = 0 \qquad (79)$$

It can be seen that we can formally write:

$$\mathbf{b} = \boldsymbol{\xi} \times \boldsymbol{\eta} \times \boldsymbol{\zeta} \times \mathbf{w} \qquad (80)$$

where: $\mathbf{b} = \left[b_1, b_2, b_3, b_4, b_5\right]^T$ $\boldsymbol{\xi} = \left[x_1, x_2, x_3, x_4, x\ \right]^T$

$\boldsymbol{\eta} = \left[y_1, y_2, y_3, y_4, y\ \right]^T$ $\boldsymbol{\zeta} = \left[z_1, z_2, z_3, z_4, z\ \right]^T$

$\mathbf{w} = \left[w_1, w_2, w_3, w_4, w\ \right]^T$

The conditions – if the point is inside the given triangle – are slightly more complex and the condition $0 \le a_i \le 1 \ \ i = 1,...,4$ can be expressed by the following criteria:

$$\begin{aligned} 0 &\le \left(-b_1 : w_2 w_3 w_4 w\ \right) \le 1 \\ 0 &\le \left(-b_2 : w_1 w_3 w_4 w\ \right) \le 1 \\ 0 &\le \left(-b_3 : w_1 w_2 w_4 w\ \right) \le 1 \\ 0 &\le \left(-b_4 : w_1 w_2 w_3 w\ \right) \le 1 \end{aligned} \qquad (81)$$

It is worth noting that the equations for the computation of barycentric coordinates given above can be simplified for special cases, e.g. if the tetrahedron vertices are expressed in the Euclidean coordinates or the given point **x** is expressed in the Euclidean coordinates. Such simplifications will increase the speed of computation significantly without compromising the robustness of the computation. Nevertheless, the resulting barycentric coordinates are generally in the projective space, i.e. the homogeneous coordinate is not equal to '1' in general.

6. Conclusion

This paper presents a new approach computation of:

- a line in the E^2 and a plane in the E^3,
- an intersection point of two lines in the E^2 and three planes in the E^3,
- a parametric equation of a line in the E^3 if given by two points or two planes in the E^3,
- barycentric coordinates

using the homogeneous coordinates directly has been presented. The presented approach enables a unified solution for the case when the line is given by two points in E^3 and also as an intersection of two planes in homogeneous coordinates directly. The presented approach uses the projective space, the principle of duality and gives several advantages over the known approaches like robustness, and avoiding division operation. It also simplifies some algorithms, e.g. line clipping and offers additional algorithms speed-ups [9] and new formula developments [10].

There is a hope that the Plücker coordinates and the projective space representation will be useful for development of new methods based on intersection computation and will allow derivation of robust algorithms with higher efficiency.

This paper also describes a robust barycentric coordinates computation for a triangle and for a tetrahedron in the Euclidean and projective spaces. The presented approach can be applied not only to the tests "point inside …" but also for the line/ray intersection problems as well. The barycentric coordinates can be computed **without a division operation**; the use of a division operation is postponed for the final evaluation step if necessary.

If the point, for which barycentric coordinates are computed, or the vertices of a triangle or a tetrahedron are given in the homogeneous coordinates, no division by a homogeneous coordinate is needed and the computation is done directly using homogeneous coordinates.

The computation of barycentric coordinates is more robust especially for thin triangles or tetrahedra as we do not use a division operation that causes instability and decreases the robustness of the computation in general.

There is some hope that this approach could help to solve certain problems with the robustness and instability of some algorithms in specific cases, too.

The presented approach can also be applied to an effective use of GPU as instead of solving a linear system of equations a cross product operation can be used, see [24].

The principle of duality and the use of homogeneous coordinates can lead to new directions in the design of algorithms, leading to simple, robust and faster algorithms, e.g. the line clipping algorithm in E^2 [9], [18], polygon or polyhedron intersection tests [23], [20], [21], [25].

Appendix A presents a simple sequence of the cross product computation in Cg/HLSL using PLib[24].

Many interesting hints for more general approach can be found in [11], [12].

7. Acknowledgments

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Appendix A

The following formula for finding the intersection point of three planes in the E^3 can be found [13]:

$$\boldsymbol{X} = \frac{D_1(\boldsymbol{n}_2 \times \boldsymbol{n}_3) + D_2(\boldsymbol{n}_3 \times \boldsymbol{n}_1) + D_3(\boldsymbol{n}_1 \times \boldsymbol{n}_2)}{\boldsymbol{n}_1(\boldsymbol{n}_2 \times \boldsymbol{n}_3)} \tag{A.1}$$

where: $D_i = \mathbf{n}_i^T \mathbf{X}$ and $\boldsymbol{X} = [X, Y, Z]^{\mathrm{T}}$

It is obvious that the notation is not only difficult to remember, but also it is “invisible” how the formula was derived.

Appendix B

There is a double cross-product used in deriving Eq.34 from Eq.33. Let us review the double cross-product equality.

Definition
Let **a**, **b** and **c** are vectors. Then:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \,.\, \boldsymbol{c})\,.\,\boldsymbol{b} - (\boldsymbol{a} \,.\, \boldsymbol{b})\,.\,\boldsymbol{c} = (\boldsymbol{a}^T \boldsymbol{c})\,.\,\boldsymbol{b} - (\boldsymbol{a}^T \boldsymbol{b})\,.\,\boldsymbol{c} \tag{A.2}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{c} \,.\, \boldsymbol{a})\,.\,\boldsymbol{b} - (\boldsymbol{c} \,.\, \boldsymbol{b})\,.\,\boldsymbol{a} = (\boldsymbol{c}^T \boldsymbol{a})\,.\,\boldsymbol{b} - (\boldsymbol{c}^T \boldsymbol{b})\,.\,\boldsymbol{a}$$

where: “$\boldsymbol{a} \,.\, \boldsymbol{b}$” means the “dot-product” of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equivalent to scalar multiplication and $(\ )^{\mathrm{T}}$ means a vector transposition.

Let us reconsider the Eq.33:

$$\boldsymbol{\omega} \times \boldsymbol{q} = \boldsymbol{v} \tag{A.3}$$

and the line $\boldsymbol{q}(t)$ equation:

$$\boldsymbol{q}(t) = \boldsymbol{q}_1 + \omega\, t \tag{A.4}$$

Then the left hand side of Eq.A.3 multiplied by ω from the right:

$$(\omega \times \boldsymbol{q}) \times \omega = (\omega \,.\, \omega).\, \boldsymbol{q} - (\omega \,.\, \boldsymbol{q}).\, \omega = \|\, \omega \,\|^2 \boldsymbol{q} - 0 = \|\, \omega \,\|^2 \boldsymbol{q} \tag{A.5}$$

It can be seen that the vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}$ are orthogonal, see Eq.33, i.e. $\omega \,.\, \boldsymbol{v} = 0$, and therefore the Eq.46 becomes:

$$\|\, \omega \,\|^2 \boldsymbol{q} = \boldsymbol{v} \times \omega \tag{A.6}$$

This equality must be valid also for the point $\boldsymbol{q}_1$ and therefore:

$$\|\, \omega \,\|^2 \boldsymbol{q}_1 = \boldsymbol{v} \times \omega \tag{A.7}$$

and if $\|\, \omega \,\|^2 \neq 0$ we can write:

$$\boldsymbol{q}_1 = \boldsymbol{v} \times \omega \,/\, \|\, \omega \,\|^2 \tag{A.8}$$

Substituting (A.8) to (A.4) we obtain:

$$\boldsymbol{q}(t) = \boldsymbol{v} \times \omega \,/\, \|\, \omega \,\|^2 + \omega\, t \tag{A.9}$$

that is identical with an Eq.34.

Appendix C

The cross product in 4D is defined as

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 \times \boldsymbol{x}_3 = \det \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix} \tag{A.10}$$

and can be implemented in Cg/HLSL on a GPU as follows:

```
float4 cross_4D(float4 x1, float4 x2, float4 x3)
{
    float4 a;
    a.x = dot(x1.yzw, cross(x2.yzw, x3.yzw));
    a.y = - dot(x1.xzw, cross(x2.xzw, x3.xzw));
    // or a.y = dot(x1.xzw, cross(x3.xzw, x2.xzw));
    a.z = dot(x1.xyw, cross(x2.xyw, x3.xyw));
    a.w = - dot(x1.xyz, cross(x2.xyz, x3.xyz));
    // or a.w = dot(x1.xyz, cross(x3.xyz, x2.xyz));

    return a;
}
```

or more compactly

```
float4 cross_4D(float4 x1, float4 x2, float4 x3)
{
    return ( dot(x1.yzw, cross(x2.yzw, x3.yzw)),
    - dot(x1.xzw, cross(x2.xzw, x3.xzw)),
    dot(x1.xyw, cross(x2.xyw, x3.xyw)),
    - dot(x1.xyz, cross(x2.xyz, x3.xyz)) );
}
```

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