Reconsidering and Rethinking Quaternionic Special Relativity

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ABSTRACT

Special relativity can be modelled mathematically with complex quaternions. The relation between this quaternionic special relativity and spacetime algebra will be discussed from a didactical perspective showing the intrinsic relations between quaternion matrices, Pauli matrices, and Dirac matrices.

Keywords

Special relativity, geometric algebra, Pauli algebra, Dirac algebra, quaternions, physics education.

1. INTRODUCTION

After having taught at several schools I returned again to academic life and university some years ago. Knowing nothing about geometric algebra and spacetime algebra at this time one of the ideas I followed was to implement quaternions in spacetime. The didactical path I chose was simple: The Lorentz transformation is a four-dimensional rotation of space and time, and quaternions can be used to describe rotations mathematically in a very elegant manner [Hor02]. Thus I wanted to find a didactical convincing way of modelling Lorentz transformations with quaternions.

Then, years later, I heard about geometric algebra and the rich didactical possibilities it opens. I was immediately overwhelmed by the clear and precise way geometrical meaning is given to algebraic expressions in geometric algebra. And at the same time the profound algebraic meaning appearing behind geometric objects, which can be described algebraically in a clear, precise, and simple way, convinced me [Hor07]. Therefore I nearly at once changed my didactical direction and modelled the Lorentz transformation in spacetime algebra [Hor09], which is the four-dimensional extension of three-dimensional geometric algebra.

It is now time to connect the two loose ends of quaternion algebra and geometric algebra with respect to special relativity and to find a didactical bridge from one system to the other.

2. QUATERNIONIC SPECIAL RELATIVITY

Of course I was not the first who tried to formulate special relativistic relations using quaternions. Among others Silberstein [Sil12], [Sil14] Conway [Con47], Lanczos [Lan19], Flint [Fli20] and Blaton [Bla35] were the first who succeeded in doing so. A well-written overview about the historical development of special relativity and the relation to the mathematics of quaternions can be found in Klein (1927/ 1928) [Kle79], presenting his lectures, which were given before the invention of Pauli matrices in 1927 [Pau27].

In these early quaternionic special relativistic papers every quaternionic basic unit $\vec{i}$, $\overleftarrow{j}$, and $\vec{k}$ is then interpreted as basic unit vector in x-, y-, and z-direction. The vector $\overleftarrow{r}$ can then be written as linear combination

$$\overleftarrow{r} = ct\,\vec{1} + ix\,\vec{i} + iy\,\overleftarrow{j} + iz\,\vec{k} \tag{1}$$

with $$\vec{i}^{\,2} = \overleftarrow{j}^{\,2} = \vec{k}^{\,2} = -\vec{1} \tag{2}$$

and $$\begin{aligned} \vec{i}\,\overleftarrow{j} &= -\overleftarrow{j}\,\vec{i} = \vec{k} \\ \overleftarrow{j}\,\vec{k} &= -\vec{k}\,\overleftarrow{j} = \vec{i} \\ \vec{k}\,\vec{i} &= -\vec{i}\,\vec{k} = \overleftarrow{j} \end{aligned} \tag{3}$$

The arrows indicate that in this style of writing these objects are indeed thought as vectors [Hor02]. For a special reason some of the arrows here show to opposite directions.

The basic unit vector $\vec{1}$ with

$$\vec{1}^2 = \vec{1} \tag{4}$$

is interpreted as vector in the direction of time which is commutative with respect to every other mathematical object.

A Lorentz transformation is then normally given as spacetime rotation

$$\vec{r}_{rot} = \vec{q}\ \vec{r}\ \vec{q}^{\,*} \tag{5}$$

or

$$\vec{r}_{rot} = \vec{q}\ \vec{r}\ \vec{q}^{\,\sim} \tag{6}$$

with the unit quaternion

$$\vec{q} = q_t\ \vec{1} + iq_x\ \vec{i} + iq_y\ \overleftarrow{j} + iq_z\ \vec{k} \tag{7}$$

Which transformation formula (5) resp. (6) should be used depends on how these vectors are defined. They have to mirror the structure of space and time mathematically. Although nature seems to be unique, there are many different mathematical mirrors which can be used to describe our world.

3. SPECIAL RELATIVITY IN SPACETIME ALGEBRA

In spacetime algebra the Dirac matrices γ_t, γ_x, γ_y, and γ_z with

$$\gamma_t^2 = -\gamma_x^2 = -\gamma_y^2 = -\gamma_z^2 = 1 \tag{8}$$

and

$$\begin{aligned} \gamma_x\,\gamma_y &= -\gamma_y\,\gamma_x \qquad & \gamma_t\,\gamma_x &= -\gamma_x\,\gamma_t \\ \gamma_y\,\gamma_z &= -\gamma_z\,\gamma_y \qquad & \gamma_t\,\gamma_y &= -\gamma_y\,\gamma_t \\ \gamma_z\,\gamma_x &= -\gamma_x\,\gamma_z \qquad & \gamma_t\,\gamma_z &= -\gamma_z\,\gamma_t \end{aligned} \tag{9}$$

are the basic unit vectors of our four-dimensional world we live in [Dor03]. The spacetime vector $\vec{r}$ can then be written as

$$\underline{\underline{r}} = ct\ \gamma_t + x\ \gamma_x + y\ \gamma_y + z\ \gamma_z \tag{10}$$

and a Lorentz transformation is again a spacetime rotation

$$\underline{\underline{r}}_{rot} = \underline{\underline{m}}\,\underline{\underline{n}}\,\underline{\underline{r}}\,\underline{\underline{n}}\,\underline{\underline{m}} \tag{11}$$

now $\underline{\underline{n}}$ and $\underline{\underline{m}}$ being two unit reflection vectors

$$\begin{aligned} \underline{\underline{n}} &= n_t\,\gamma_t + n_x\,\gamma_x + n_y\,\gamma_y + n_z\,\gamma_z \\ \underline{\underline{m}} &= m_t\,\gamma_t + m_x\,\gamma_x + m_y\,\gamma_y + m_z\,\gamma_z \end{aligned} \tag{12}$$

Because Dirac matrices are behaving like vectors in spacetime algebra, they can be called Dirac vectors.

4. TRANSLATING BETWEEN QUATERNION ALGEBRA AND DIRAC ALGEBRA

What is now the relation between these different mathematical structures of quaternion algebra and Dirac algebra which obviously express the same physical situation? To answer this question we have to find a way to translate between these algebras.

Although (2 x 2)-matrices obscure and hide the geometrical meaning of algebraic objects, they can help us to find this translation.

Already before the invention of Pauli matrices it was well known that quaternions can be written as (2 x 2)-matrices [Kle79]. For example they are given in [Bla35, p. 344] as

$$\begin{aligned} \vec{i} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad & \overleftarrow{j} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \vec{k} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad & \vec{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{13}$$

These matrices are intrinsically connected with Pauli matrices [Pau27]

$$\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad & \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad & 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{14}$$

because the quaternion basic units are mere products of Pauli matrices:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ldots \tag{15}$$

$$\Rightarrow \qquad \begin{aligned} \sigma_x\,\sigma_y &= \vec{k} \\ \sigma_y\,\sigma_z &= \vec{i} \\ -\sigma_z\,\sigma_x &= \overleftarrow{j} \end{aligned} \tag{16}$$

At the same time Dirac vectors are intrinsically connected with Pauli matrices, because Pauli matrices are mere products of basic Dirac vectors (see for example [Dor03, eq. 5.37]:

$$\Rightarrow \qquad \begin{aligned} \gamma_x\,\gamma_t &= \sigma_x \\ \gamma_y\,\gamma_t &= \sigma_y \\ \gamma_z\,\gamma_t &= \sigma_z \end{aligned} \tag{17}$$

Now we are in a position to translate equation (1) into equation (10) and equations (5) or (6) into equation (11).

According to our translation rules (16) the position vector of equation (1)

$$\vec{r} = ct\ \vec{1} + ix\ \vec{i} + iy\ \overleftarrow{j} + iz\ \vec{k} \tag{1}$$

is equivalent to the confusing position vector in Pauli algebra

$$\underline{r}' = ct\,1 + ix\,\sigma_y\sigma_z - iy\,\sigma_z\sigma_x + iz\,\sigma_x\sigma_y \tag{18}$$

"Not surprisingly" the minus sign in front of the y-coordinate historically "was a potential source of great confusion" [Dor03, p. 34] because we have to change from a left-handed coordinate system to a right-handed coordinate system just by flipping the sign of one of the basic quaternion units, e.g.

$$-\overleftarrow{j} = \vec{j} = \sigma_z \sigma_x \qquad (19)$$

resulting in the right-handed position vector

$$\vec{r} = ct\,\vec{1} + ix\,\vec{i} + iy\,\vec{j} + iz\,\vec{k} \qquad (20)$$

and therefore

$$\underline{r} = ct\,1 + ix\,\sigma_y\sigma_z + iy\,\sigma_z\sigma_x + iz\,\sigma_x\sigma_y \qquad (21)$$

Inserting the trivector (or pseudoscalar)

$$I = \sigma_x \sigma_y \sigma_z \qquad (22)$$

with $$I^2 = \sigma_x \sigma_y \sigma_z\, \sigma_x \sigma_y \sigma_z = -1 \qquad (23)$$

as complex unit $i = I$ into (21) the Pauli position vector transforms into

$$\underline{r} = ct - x\,\sigma_x - y\,\sigma_y - z\,\sigma_z \qquad (24)$$

Obviously, this is no pure vector, but a sum of a timelike scalar ct and a one-dimensional spacelike vector part.

This mathematical object of Pauli algebra (with one underlining) can be transformed into an equivalent mathematical object of Dirac algebra (with two underlinings) using equations (17)

$$\underline{\underline{R}} = ct - x\,\gamma_x\gamma_t - y\,\gamma_y\gamma_t - z\,\gamma_z\gamma_t \qquad (25)$$

This again is no pure Dirac vector, but a sum of a scalar giving the time coordinate and a Dirac bivector. To finish the translation process, we have to multiply the basic timelike Dirac unit γ_t from the left:

$$\begin{aligned} \gamma_t\,\underline{\underline{R}} &= \gamma_t\,ct - \gamma_t\,x\,\gamma_x\gamma_t - \gamma_t\,y\,\gamma_y\gamma_t - \gamma_t\,z\,\gamma_z\gamma_t \\ &= ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z = \underline{\underline{r}} \qquad (26) \end{aligned}$$

Now we have got two isomorphic objects: The Dirac multivector $\underline{\underline{R}}$ associated with the quaternion vector $\vec{r}$ has to be multiplied with the timelike basic Dirac unit γ_t to get the Dirac vector $\underline{\underline{r}}$.

We therefore can conclude, that the original quaternion vector geometrically behaves as bivector, that is: as linear combination of oriented area elements. It's sort of a mathematical accident[1] that this Dirac bivector sometimes (e.g. in the case of special relativity) shows the expected Lorentz transformation behaviour of a spacetime vector, which is discussed in the next section.

[1] Freeman Dyson would call it a joke of nature, connected with the strange effects the imaginary unit i can produce [Dys09, p.213].

5. REFLECTIONS IN QUATERNION ALGEBRA AND DIRAC ALGEBRA

A reflection at the spacetime unit vector

$$\underline{\underline{n}} = n_t\,\gamma_t + n_x\,\gamma_x + n_y\gamma_y + n_z\,\gamma_z \qquad (27)$$

is given in Dirac algebra by

$$\underline{\underline{r}}_{ref} = \pm\,\underline{\underline{n}}\,\underline{\underline{r}}\,\underline{\underline{n}} \qquad (28)$$

while the positive sign is used when the reflection vector $\underline{\underline{n}}$ is a timelike vector $\underline{\underline{n}}^2 = 1$ and the negative sign is used when the reflection vector $\underline{\underline{n}}$ is a spacelike vector $\underline{\underline{n}}^2 = -1$.

Let's try to convert this back into quaternion algebra.

$$\begin{aligned} \gamma_t\,\underline{\underline{r}}_{ref} &= \pm\,\gamma_t\,\underline{\underline{n}}\,\underline{\underline{r}}\,\underline{\underline{n}} \\ &= \pm\,\gamma_t\,\underline{\underline{n}}\,\underline{\underline{r}}\,\gamma_t\,\gamma_t\,\underline{\underline{n}} \qquad (29) \end{aligned}$$

with

$$\begin{aligned} \underline{\underline{N}} &= \gamma_t\,\underline{\underline{n}} = n_t - n_x\,\gamma_x\gamma_t - n_y\,\gamma_y\gamma_t - n_z\,\gamma_z\gamma_t \\ \underline{\underline{R}}' &= \underline{\underline{r}}\,\gamma_t = ct + x\,\gamma_x\gamma_t + y\,\gamma_y\gamma_t + z\,\gamma_z\gamma_t \qquad (30) \end{aligned}$$

Please compare the different signs of $\underline{\underline{R}}'$ and $\underline{\underline{R}}$ in equations (25) and (30). Converting back into Pauli algebra (with one underlining) gives

$$\underline{r}_{ref} = \pm\,\underline{n}\,\underline{r}'\,\underline{n} \qquad (31)$$

with

$$\begin{aligned} \underline{n} &= n_t - n_x\,\sigma_x - n_y\,\sigma_y - n_z\,\sigma_z \\ \underline{r}' &= ct + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z \qquad (32) \end{aligned}$$

Substituting

$$\begin{aligned} i\,\sigma_y\sigma_z &= -\,\sigma_x \\ i\,\sigma_z\sigma_x &= -\,\sigma_y \qquad (33) \\ i\,\sigma_x\sigma_y &= -\,\sigma_z \end{aligned}$$

changes equations (32) into

$$\begin{aligned} \underline{n} &= n_t 1 + i n_x\,\sigma_y\sigma_z + i n_y\,\sigma_z\sigma_x + i n_z\,\sigma_x\sigma_y \\ \underline{r}' &= ct\,1 - ix\,\sigma_y\sigma_z - iy\,\sigma_z\sigma_x - iz\,\sigma_x\sigma_y \qquad (34) \end{aligned}$$

Now (31) can be translated back into quaternion algebra:

$$\vec{r}_{ref}\,' = \pm\,\vec{n}\,\vec{r}\,'\,\vec{n} \qquad (35)$$

with

$$\begin{aligned} \vec{n} &= n_t\,\vec{1} + i n_x\,\vec{i} + i n_y\,\vec{j} + i n_z\,\vec{k} \\ \vec{r}_{ref}\,' &= ct\,\vec{1} - ix\,\vec{i} - iy\,\vec{j} - iz\,\vec{k} \qquad (36) \end{aligned}$$

To get rid of the minus signs in (30) we have to change equation (35) into

$$\vec{r}_{ref} = \pm\,\vec{n}\,\vec{r}^{\,*}\,\vec{n} \qquad (37)$$

in a right-handed coordinate system. Thus it is possible to express spacetime reflections in quaternion algebra, but it is of course not as elegant as in spacetime algebra.

Examples

To illustrate the possible different cases three examples are shown below. First a pure space reflection of the spacetime vector

$$\vec{r} = ct\,\vec{1} + ix\,\vec{i} + iy\,\vec{j} + iz\,\vec{k}$$

or $\underline{\underline{r}} = ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z$

at the x-axes with reflection vector

$$\vec{n}_1 = i\,\vec{i} \qquad \text{with} \quad \vec{n}_1\,\vec{n}_1{}^* = -1$$

or $\underline{\underline{n}}_1 = \gamma_x$ with $\underline{\underline{n}}_1{}^2 = -1$

gives the reflected vector

$$\vec{r}_{1ref} = -i\vec{i}\;\vec{r}^*\;i\vec{i}$$
$$= -ct\,\vec{1} + ix\,\vec{i} - iy\,\vec{j} - iz\,\vec{k}$$

or $\underline{\underline{r}}_{1ref} = -\gamma_x\,(ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z)\,\gamma_x$

$$= -ct\,\gamma_t + x\,\gamma_x - y\,\gamma_y - z\,\gamma_z$$

with all components exchanging the sign except the x-coordinate.

Secondly, a pure space reflection of the spacetime vector $\vec{r}$ or $\underline{\underline{r}}$ at the diagonal line of the xy-plane with reflection vector

$$\vec{n}_2 = \frac{1}{\sqrt{2}}\left(i\,\vec{i} + i\,\vec{j}\right) \qquad \text{with} \quad \vec{n}_2\,\vec{n}_2{}^* = -1$$

or $\underline{\underline{n}}_2 = \frac{1}{\sqrt{2}}\left(\gamma_x + \gamma_y\right)$ with $\underline{\underline{n}}_2{}^2 = -1$

gives the reflected vector

$$\vec{r}_{2ref} = -\frac{1}{2}\left(i\,\vec{i} + i\,\vec{j}\right)\;\vec{r}^*\;\left(i\,\vec{i} + i\,\vec{j}\right)$$
$$= -ct\,\vec{1} + iy\,\vec{i} + ix\,\vec{j} - iz\,\vec{k}$$

or $\underline{\underline{r}}_{2ref} = -\frac{1}{2}\,(\gamma_x+\gamma_y)\,(ct\gamma_t+x\gamma_x+y\gamma_y+z\gamma_z)\,(\gamma_x+\gamma_y)$

$$= -ct\,\gamma_t + y\,\gamma_x + x\,\gamma_y - z\,\gamma_z$$

with negative components of time- and the z-coordinate and the components of the x- and y-directions exchanged.

As third example the spacetime reflection of the spacetime vector $\vec{r}$ or $\underline{\underline{r}}$ at the time-axes with reflection vector

$$\vec{n}_3 = \vec{1} \qquad \text{with} \quad \vec{n}_3\,\vec{n}_3{}^* = +1$$

or $\underline{\underline{n}}_1 = \gamma_t$ with $\underline{\underline{n}}_3{}^2 = +1$

is considered. Now the reflected vector

$$\vec{r}_{3ref} = \vec{1}\;\vec{r}^*\;\vec{1}$$
$$= ct\,\vec{1} - ix\,\vec{i} - iy\,\vec{j} - iz\,\vec{k}$$

or $\underline{\underline{r}}_{3ref} = \gamma_t\,(ct\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z)\,\gamma_t$

$$= ct\,\gamma_t - x\,\gamma_x - y\,\gamma_y - z\,\gamma_z$$

arises, and all components of space directions exchange the sign while the component of the time-coordinate does not change.

Finally the spacetime reflection of the spacetime vector $\vec{r}$ or $\underline{\underline{r}}$ at the timelike unit reflection vector

$$\vec{n}_4 = \frac{1}{4}\left(5\cdot\vec{1} + 3i\,\vec{i}\right) \qquad \text{with} \quad \vec{n}_4\,\vec{n}_4{}^* = +1$$

or $\underline{\underline{n}}_4 = \frac{1}{4}\left(5\,\gamma_t + 3\,\gamma_x\right)$ with $\underline{\underline{n}}_4{}^2 = +1$

gives the reflected vector

$$\vec{r}_{4ref} = \frac{1}{16}\left(5\cdot\vec{1} + 3i\,\vec{i}\right)\;\vec{r}^*\;\left(5\cdot\vec{1} + 3i\,\vec{i}\right)$$
$$= \frac{34ct - 30x}{16}\,\vec{1} + i\frac{30ct - 34x}{16}\,\vec{i} - iy\,\vec{j} - iz\,\vec{k}$$

or $\underline{\underline{r}}_{4ref} = \frac{1}{16}\,(5\gamma_t+3\gamma_x)\,(ct\gamma_t+x\gamma_x+y\gamma_y+z\gamma_z)\,(5\gamma_t+3\gamma_x)$

$$= \frac{34\,ct - 30\,x}{16}\gamma_t + \frac{30\,ct - 34\,x}{16}\gamma_x - y\,\gamma_y - z\,\gamma_z$$
$$= (2{,}125\,ct - 1{,}875\,x)\,\gamma_t$$
$$+ (1{,}875\,ct - 2{,}125\,x)\,\gamma_x - y\,\gamma_y - z\,\gamma_z$$

which can be checked by squaring:

$$\vec{r}_{4ref}\;\vec{r}_{4ref}{}^* = \vec{r}\;\vec{r}^* = (ct)^2 - x^2 - y^2 - z^2$$
$$\underline{\underline{r}}_{4ref}{}^2 = \underline{\underline{r}}^2 = ct^2 - x^2 - y^2 - z^2$$

6. ROTATIONS IN QUATERNION ALGEBRA AND DIRAC ALGEBRA

A succession of two reflections gives always a rotation, and every rotation can be decomposed into two (or four or six or any other even number of) reflections.

Therefore a reflection of the reflected Dirac vector $\underline{\underline{r}}_{ref}$ at the spacetime unit vector

$$\underline{\underline{m}} = m_t\,\gamma_t + m_x\,\gamma_x + m_y\,\gamma_y + m_z\,\gamma_z \qquad (38)$$

is given in Dirac algebra by

$$\underline{\underline{r}}_{rot} = \pm\,\underline{\underline{m}}\;\underline{\underline{r}}_{ref}\;\underline{\underline{m}} \qquad (39)$$

and leads to a spacetime rotation. As usual the positive sign is used when the reflection vector $\underline{\underline{m}}$ is a timelike vector $\underline{\underline{m}}^2 = 1$ and the negative sign is used when the reflection vector $\underline{\underline{m}}$ is a spacelike vector $\underline{\underline{m}}^2 = -1$.

When both reflection vectors are of equal quality (both spacelike or both timelike) the signs cancel

$$\underline{\underline{r}}_{rot} = \pm\,\underline{\underline{m}}\;(\pm\,\underline{\underline{n}}\;\underline{\underline{r}}\;\underline{\underline{n}})\;\underline{\underline{m}}$$
$$= \underline{\underline{m}}\;\underline{\underline{n}}\;\underline{\underline{r}}\;\underline{\underline{n}}\;\underline{\underline{m}} \qquad (40)$$

Let's try again to convert this back into quaternion algebra.

$$\gamma_t\,\underline{\underline{r}}_{rot} = \gamma_t\,\underline{\underline{m}}\;\underline{\underline{n}}\;\underline{\underline{r}}\;\underline{\underline{n}}\;\underline{\underline{m}}$$
$$= \gamma_t\,\underline{\underline{m}}\;\underline{\underline{n}}\;\gamma_t\,\gamma_t\,\underline{\underline{r}}\;\underline{\underline{m}}\;\gamma_t\,\gamma_t\,\underline{\underline{n}} \qquad (41)$$

This time we do not have to care about the signs of the original vector $\underline{\underline{R}} = \gamma_t\,\underline{\underline{r}}$, but of the signs of the unit rotation quaternions, as two successive applications of equation (37) show:

$$\begin{aligned}\vec r_{rot} &= \pm\, \vec m\, \vec r_{ref} * \vec m \\ &= \pm\, \vec m\, (\pm \vec n\, \vec r * \vec n) * \vec m \\ &= \vec m\, \vec n * \vec r\, \vec n * \vec m\end{aligned} \tag{42}$$

Consistently we have to use equation (6), and the unit rotation quaternions become

$$\vec q = \vec m\, \vec n^{*}$$

and

$$\vec q^{\sim} = \vec n^{*}\, \vec m \tag{43}$$

in a right-handed coordinate system.

Examples

The differences and similarities between quaternionic rotations and Dirac spacetime rotations will be illustrated in the following. First we reflect the time-like Point A (symbolised by the vector $\vec r_A$ or $\underline{r}_A$, which lies inside the future light cone of an observer at the origin of the coordinate system) and the light-like point B (symbolised by the vector $\vec r_B$ or $\underline{r}_B$, which lies on the future light cone)

$$\vec r_A = 5 \cdot \vec 1 + 4i\, \vec\imath$$

$$\vec r_B = 4 \cdot \vec 1 + 4i\, \vec\imath$$

or

$$\underline{r}_A = 5\, \gamma_t + 4\, \gamma_x$$

$$\underline{r}_B = 4\, \gamma_t + 4\, \gamma_x$$

at the unit reflection vectors

$$\vec n = \frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 + i\, \vec\imath\right)$$

$$\vec m = \vec 1$$

or

$$\underline{\underline{n}} = \frac{1}{\sqrt{8}}\left(3\,\gamma_t + \gamma_x\right)$$

$$\underline{\underline{m}} = \gamma_t$$

with

$$\vec n\, \vec n^{*} = \vec m\, \vec m^{*} = +1$$

$$\underline{\underline{n}}^2 = \underline{\underline{m}}^2 = +1$$

The reflected position vectors then are

$$\vec r_{Aref} = 3{,}25 \cdot \vec 1 - 1{,}25 i\, \vec\imath$$

$$\vec r_{Bref} = 2 \cdot \vec 1 - 2i\, \vec\imath$$

or

$$\underline{r}_{Aref} = 3{,}25\, \gamma_t - 1{,}25\, \gamma_x$$

$$\underline{r}_{Bref} = 2\, \gamma_t - 2\, \gamma_x$$

With $\vec q = \frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 - i\, \vec\imath\right)$ and

$$\vec q^{\sim} = \frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 - i\, \vec\imath\right)$$

the rotated Dirac position vectors become

$$\begin{aligned}\vec r_{Arot} &= \frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 - i\vec\imath\right)\left(5 \cdot \vec 1 + 4i\, \vec\imath\right)\frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 - i\vec\imath\right) \\ &= \frac{1}{8}\left(11 \cdot \vec 1 + 7i\vec\imath\right)\left(3 \cdot \vec 1 - i\vec\imath\right) \\ &= 3{,}25 \cdot \vec 1 + 1{,}25 i\vec\imath\end{aligned}$$

$$\begin{aligned}\vec r_{Brot} &= \frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 - i\vec\imath\right)\left(4 \cdot \vec 1 + 4i\, \vec\imath\right)\frac{1}{\sqrt{8}}\left(3 \cdot \vec 1 - i\vec\imath\right) \\ &= \frac{1}{8}\left(8 \cdot \vec 1 + 8i\vec\imath\right)\left(3 \cdot \vec 1 - i\vec\imath\right) \\ &= 2 \cdot \vec 1 + 2i\vec\imath\end{aligned}$$

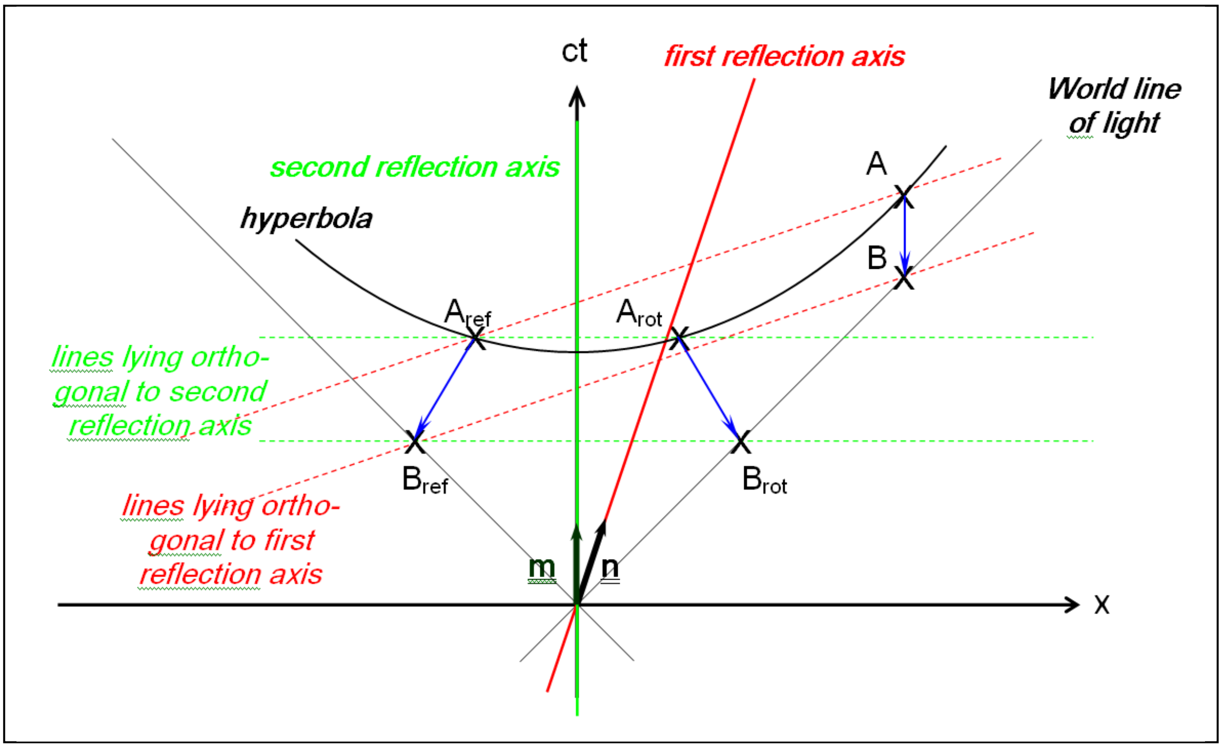

Figure 1. Construction of a spacetime rotation of two points A and B by two successive spacetime reflections at the first (red) and second (green) reflection axes.

or $$\underline{r}_{Arot} = \frac{1}{\sqrt{8}}(3-\gamma_x\gamma_t)(5\gamma_t+4\gamma_x)\frac{1}{\sqrt{8}}(3+\gamma_x\gamma_t)$$
$$= \frac{1}{8}(11\gamma_t + 7\gamma_x)(3+\gamma_x\gamma_t)$$
$$= 3{,}25\ \gamma_t + 1{,}25\ \gamma_x$$
$$\underline{r}_{Brot} = \frac{1}{\sqrt{8}}(3-\gamma_x\gamma_t)(4\gamma_t+4\gamma_x)\frac{1}{\sqrt{8}}(3+\gamma_x\gamma_t)$$
$$= \frac{1}{8}(8\gamma_t + 8\gamma_x)(3+\gamma_x\gamma_t)$$
$$= 2\ \gamma_t + 2\ \gamma_x$$

(see figure 1). The original and the transformed points B always lie on the world line of light, as lightlike vectors remain lightlike when reflected or rotated.

Thus both mathematical concepts describe Lorentz transformation appropriate and correct, and uncover the geometrical meaning of this transformation.

7. METACONCEPTUAL AWARENESS IN PHYSICS AND MATHEMATICS

Quaternions played a major role in the historical development of mathematics and physics. When lecturing about this development Felix Klein emphasized the importance of Hamilton's and Grassmann's ideas several times [Kle79]. Analyzing the heuristic role of quaternions Anderson and Joshi thus concluded, that "unique features of quaternionic structures have been woven closely into the development of new physical theories" [And02, p. 15].

These intrinsic relationship was already discussed by Grassmann [Gra77], who connected his theory of extensions with the theory of quaternions formulated nearly at the same time by Hamilton. Quaternions and geometric algebra give us two different mathematical views on our world. Obviously this world outside us is (as far as we know) unique, one and single, but we do not have a unique and single way to describe this world mathematically.

We posses lots of different mathematical languages (or as Hestenes once said: "a Babel of mathematical tongues" [Hes03, p. 106]). But we do not know which mathematical language will be appropriate to solve the physics problems of the future. We only know which language today fits best to solve our contemporary problems. Therefore it is our task not only to teach and explain geometric or spacetime algebra, but to present and discuss the position geometric algebra takes by exploring the relation to other and different mathematical concepts. Let it never happen again that a prominent scientist has to say with respect to the relation between the Dirac equation and geometry: "Had we been better educated in physics, or had there been the kind of dialogue with physicists that is now common, we would have got there much sooner" [Ati08, p. 116].

For this reason we should pay attention to developing meta-conceptual awareness when teaching physics and mathematics or other fields of science. Our students should be able to change mathematical and physical perspectives to look at problems from different angles and to analyze and explain problems in distinct ways.

As a research scientist it might be frustrating that "much time (is) being taken up with mere translation between the two modes of expression" [Hes71, p. 1013]. But as a teacher or lecturer of physics it is necessary to push and to urge our students to think through these translations. One of the most promising ways to implement (that is to learn) knew knowledge is to create cognitive conflicts between different perspectives to let the students find a solution of these conflicts. We would be bad teachers of physics and mathematics if we presented one and only one truth.

As a final remark I want to look back into the history of mathematics and especially into the history of the quaternionists who wanted to promote quaternion algebra – and who failed. We can learn something about this failure. Felix Klein and Arnold Sommerfeld used the theory of quaternions in a convincing and impressive way to explain the physics of the gyroscope (see section 7 of chapter I in [Kle97]). At the end of this chapter Klein and Sommerfeld wrote: "...we want to bring forward also an advantage which is as well attributed to the theory of quaternions and vector analysis, namely the independence of their operations and their basic units from the coordinate system. However ... it would mean to misunderstand the character of analytic geometry if we always and in principle would not use coordinates explicitly when performing calculations." [Kle97, p.68].

Klein and Sommerfeld here describe the struggle between supporters of coordinate free methods in quaternion algebra and the supporters of a more frequent use of coordinates. Their conclusion: "It is important to think invariantly, not to calculate invariantly."

In geometric algebra we are in a similar situation today, when supporters of coordinate free methods claim that essential calculation should be done without coordinates. Is this really possible? Goldman gives a clear answer: "Does the full geometric algebra really lead to coordinate free methods for all of Computer Graphics? Again in my experience the answer is no" [Gol08, p. 656].

We should therefore show a meta-conceptual openness and teach both strategies: working with and working without coordinates. The invention of coordinates was at least a decisive step in the conceptual development of modern mathematics, as Freeman

Dyson writes: "It means that the deepest concepts in mathematics are those which link one world of ideas with another. In the seventeenth century Descartes linked the disparate worlds of algebra and geometry with his concept of coordinates" [Dys09, p. 218]. In a similar way geometric algebra works: It again links the disparate worlds of algebra and geometry.

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