Using Geometric Algebra for Visualizing Integral Curves

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ABSTRACT

The Differential Geometry of curves is described by means of the Frenet-Serret formulas, which cast first, second and third order derivatives into curvature and torsion. While in usual vector calculus these quantities are usually considered to be scalar values, formulating the Frenet-Serret equations in the framework of Geometric Algebra exhibits that they are best described by a bivector for the curvature and a trivector for the torsion. The bivector curvature field is directly suitable for visualization of integral curves for vector fields, providing "Frenet Ribbons" which are much richer in their visual expressiveness than lines. The set of quantities in the Frenet-Serret formalism allows to study numerical pitfalls for computing Frenet Ribbons. We show how to address them and demonstrate the applicability of the technique upon a complex numerical data set from computational fluid dynamics.

Keywords: Frenet Ribbon, pathline, streamline, computational fluid dynamics, curvature, torsion

1 INTRODUCTION

Numerical algorithms ultimately need to work with coordinates in the form of real numbers, thus $\mathbb{R}^n$. However, the early introduction of coordinates in the mathematical formulation of algorithms is, though common practice, highly problematic, as it obscures the view to the actual mathematical properties of the involved objects. Once an abstract mathematical object has been dismantled into numbers, even simple properties become complex. For instance, Sethian formulates the issue as "*the use of a coordinate system has nothing to do with the problem, but it has severely constrained our options*" [8] and Hermann Weyl wrote "*The introduction of numbers as coordinates by reference to the particular division scheme of the one-dimensional open continuum is an act of violence [...]*"[12]. In other words: while the algebraic operations on real numbers, the one-dimensional space $\mathbb{R}$, are part of common knowledge, it is a severe and unnecessary restriction to reduce other spaces – in particular n-dimensional manifolds such as used in geometry – to this set of one-dimensional algebraic operations (which is the process of introducing coordinates). Rather n-dimensional manifolds may carry an algebraic structure by themselves which should just be used, such as demonstrated by the coordinate-free approach of Geometric Algebra (GA) [2] which can directly be implemented using programming languages such as C++.

Curvature and torsion are common measures to express the properties of curves [7], as they represent the second and third derivative. In scientific visualization these measures serve to analyze the properties of streamlines in vector fields, but they can also be determined directly from the vector field itself, bypassing the process of computing an integral line [11].

Curvature and torsion are commonly introduced using vector algebra in the Euclidean space, involving formulations based on the cross-product. This formulation has several drawbacks: it hides the fact that torsion is a signed quantity, changing the sign under reflection; it relies on Cartesian coordinates, obscuring how to compute torsion with an explicit metric tensor such as required for curved space in relativity or curvilinear coordinates in computational fluid dynamics, and last not least formulations based on the cross-product do not generalize to higher dimension, which is required when we want to extend the formalism to study time-dependent vector fields in a four-dimensional framework. While there are n-dimensional vector calculus formulations of the Frenet-Serret formulae [7] available, GA improves the intuition of the involved objects.

The Frenet-Serret apparatus fails when the curvature becomes zero. Extensions to the Frenet frame have been proposed using quaternion formulations [6, 5]. In this paper we restrict ourselves to a review of the Frenet frame enhanced by a coordinate-free geometrical interpretation which is more intuitive than quaternions or vectors. The presented calculus is independent from the dimension of the underlying manifold and is expected to generalize to higher dimensions and treatment of parallel transport more intuitively in future work.

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2 MATHEMATICAL BACKGROUND

2.1 Differential Geometry

An n-dimensional manifold M is a topological space that locally looks like $\mathbb{R}^n$. For each point there exists a neighborhood and a mapping, called a *chart* $\{x^\mu\}: M \to \mathbb{R}^n$. The transition from one chart $\{x^\mu\}$ to another chart $\{x^{\bar{\mu}}\}$ defines n coordinate transformation functions $\{x^\mu\}\left(\{x^{\bar{\mu}}\}^{-1}\right): X \to Y$ with $X,Y \subset \mathbb{R}^n$.If these are differentiable k times, then the manifold is said to be $\mathscr{C}^k$. Space and time is modeled in physics as a $\mathscr{C}^\infty$-manifold. The laws of physics are independent of the choice of a coordinate system, which provide just representations of the mathematical objects.

Curves vs. Lines A *curve* is a mapping from a scalar $\lambda \in \mathbb{R}$, the curve parameter, to a point on a manifold: $q: \mathbb{R} \to M: \lambda \mapsto q(\lambda)$. The image of q in M is a *line*, a one-dimensional manifold. A certain line can be described by many curves which are distinct by different parameterizations. To describe a curve in a certain chart $\{x^\mu\}$ a coordinate function is used to extract a function for each coordinate of the chart:

$$\begin{array}{rcl} q^\mu: \mathbb{R} & \to & \mathbb{R} \\ \lambda & \to & q^\mu(\lambda) = x^\mu(q(\lambda)) \equiv x^\mu \circ q(\lambda) \end{array} \tag{1}$$

This set of n functions $q^\mu(\lambda)$ is the representation of the curve q in a coordinate system.

Tangential Vectors A tangential vector v may be understood as a small displacement of neighboring points on a curve, where the components of the tangent vector are given by $v^\mu = dq^\mu(\lambda)/d\lambda$. Given a differentiable function $f: M \to \mathbb{R}$ we may evaluate it along the curve $f(q(\lambda)): \mathbb{R} \to \mathbb{R}$ and find the directional derivative of f along q in a chart $\{x^\mu\}$ as

$$\frac{d}{d\lambda} f(q(\lambda)) = \frac{dq^\mu}{d\lambda} \frac{\partial f}{\partial x^\mu} \tag{2}$$

We may thus identify the tangential vector v with the derivation operation $d/d\lambda$, which is an interpretation independent of any coordinate system:

$$v \equiv \frac{d}{d\lambda} = \frac{dq^\mu}{d\lambda} \frac{\partial}{\partial x^\mu} =: q'(\lambda) \tag{3}$$

The set of all derivatives $\frac{\partial}{\partial x^\mu}$ in all directions at a point p defines the tangential space $T_p(M)$, which is a vector space (in contrast to M, which in general is not).

Wedge (Outer) Product The wedge product is denoted by the symbol "$\wedge$" and was introduced by Hermann Grassmann in the 19^{th} century. It allows to construct a vector space $\Lambda^2(T_p)$ from the tangential space T_p by introducing the anti-symmetric (or wedge) product $\wedge: T_p \times T_p \to \Lambda^2(T_p)$ with $u,v \in T_p$, whereby $u \wedge v = -v \wedge u$. Higher orders of the form $\Lambda^k(T_p)$ with $k \leq dim(M)$ consist of so called *k-vectors* with *scalars* as *0-vectors*, *vectors* as *1-vectors*, *bivectors* as *2-vectors*, and so forth.

Dot (Inner) Product The *metric tensor field* is a scalar-valued symmetric bilinear function g operating on tangential vectors, given at each point $p \in M$:

$$g: T_p(M) \times T_p(M) \to \mathbb{R}: u,v \mapsto g_p(u,v) \tag{4}$$

The metric tensor field allows to define the inner ("dot") product $u \cdot v := g_p(u,v)$ of two tangential vectors. The dot symbol "$\cdot$" is used by convention as a shortcut, but implying involvement of the metric tensor which needs to be explicitly specified for a manifold. In contrast, the wedge product is defined on the tangential space without any additional structure.

Arc Length and Curve Tangent Vector Arc length s is defined as the length of integrated curve segments $s(\lambda) := \int_0^\lambda |q'(\tilde{\lambda})| \, d\tilde{\lambda}$ with $|v| = \sqrt{(g_p(v,v))}$. Derivation by the arc length will be denoted by dots:

$$\dot{v} := \frac{d}{ds} v \equiv \frac{d\lambda}{ds} \frac{d}{d\lambda} v = \dot{\lambda}\, v' \quad . \tag{5}$$

In general, derivation along a curve requires to employ a tangential transport and affine connection ∇. It follows from (2.1) by derivation on both sides: $\frac{ds}{d\lambda} = |q'|$, which allows to express the derivation by arc length s via the derivation by the curve parameter:

$$\dot{v} = \frac{1}{|q'|} v' \quad \text{or} \quad \frac{d}{ds} = \frac{d\lambda}{ds} \frac{d}{d\lambda} = \frac{1}{|q'|} \frac{d}{d\lambda} \tag{6}$$

Specifically it follows that the tangential vector $\dot{q}$ with respect to arc length - defined as the *tangent vector* t - is a unit vector $|\dot{q}| = 1$ due to $t := \dot{q} = \frac{1}{|q'|} q'$.

2.2 Geometric Algebra

Geometric Algebra is the generalization of vector calculus to form a complete set of algebraic operations on tangential vectors and k-vectors. Its central concept is the introduction of the invertible *geometric product*. Given two (tangential) vectors u,v and a metric g, the requirements on the geometric product uv are to be associative, left- and right-distributive and to fulfill $u^2 = uu = g(u,u)$. These postulates lead to the geometric product as $uv = u \cdot v + u \wedge v$, which is now invertible.It is important to keep in mind that the geometric product is not commutative, thus in general $uv \neq vu$ and one needs to distinguish among left- and right-multiplication. As the geometric product sums a scalar value and a bivector it operates no longer on tangential vectors alone, but on the 2^n-dimensional space of *multivectors* $V \in \bigoplus_{k=0}^n \Lambda^k(T_p)$

Inner and Outer Product in GA Expressing the inner and outer product via the geometric product may well lead to easier expressions since the geometric product is invertible and associative. For 1-vectors the inner product is given by the symmetric part of the geometric product, the outer product as the anti-symmetric part:

$$u \cdot v \equiv \frac{1}{2}(uv + vu) \quad , \quad u \wedge v \equiv \frac{1}{2}(uv - vu) \quad . \tag{7}$$

Another useful operator, the Hodge-star operator $\star$, maps k-multivectors to $(n-k)$-vectors via the product with a pseudoscalar (an n-multivector) $\Omega \in \Lambda^n(T_p)$.

$$\star : \Lambda^k(T_p) \to \Lambda^{n-k}(T_p) : V \mapsto \Omega V \quad . \tag{8}$$

It allows to identify vectors and bivectors in three-dimensional space. For instance, the cross product in three-dimensional vector calculus corresponds to

$$u \times v \equiv \star(u \wedge v) \quad , \tag{9}$$

the difference being that "$\times$" is only defined in 3D, whereas the right side works in arbitrary dimensions.

Vector Projections Using the geometric product on two arbitrary vectors u, v the expression wuw with a unit vector $w = v/|v|$ yields the vector u as *reflected* at the vector v. Adding the reflected vector wuw to u yields the component of the vector u that is parallel to w:

$$u_{\parallel v} = \frac{1}{2}\left(u + \frac{vuv}{|v|^2}\right) \quad , \tag{10}$$

while subtraction yields the perpendicular component

$$u_{\perp v} = \frac{1}{2}\left(u - \frac{vuv}{|v|^2}\right) \quad , \tag{11}$$

where evidently $u = u_{\parallel} + u_{\perp}$. In GA (11) is called a *rejection* operation. Both components correspond to the inner and outer product (7) when multiplied with the inverse vector $v^{-1} \equiv v/|v|^2$:

$$u_{\parallel v} = \frac{(uv + vu)}{2} \, \frac{v}{|v|^2} = (u \cdot v)v^{-1} = v^{-1}(u \cdot v) \quad , \tag{12}$$

$$u_{\perp v} = \frac{(uv - vu)}{2} \, \frac{v}{|v|^2} = (u \wedge v)v^{-1} = -v^{-1}(u \wedge v) \quad . \tag{13}$$

Relation to Vector Calculus In 3D Euclidean space, we get the orthogonal component via the cross-product:

$$u_{\perp v} = \frac{v \times (u \times v)}{v^2} \tag{14}$$

Using the vector triple product formula relating cross and dot product $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$ we see

$$u_{\perp v} = \frac{u(v \cdot v) - v(u \cdot v)}{v^2} = u - (u \cdot v)v/v^2 \equiv u - u_{\parallel v}. \tag{15}$$

Derivative of a Unit Vector The derivative $d/d\lambda$, denoted by a prime as shortcut in the following expressions, of an (arbitrary) unit vector field $v/|v|$ along a curve yields a vector field that is orthogonal to the original vector field v:

$$\frac{d}{d\lambda}\frac{v}{|v|} = \frac{|v|\frac{d}{d\lambda}v - v\frac{d}{d\lambda}|v|}{|v|^2} \tag{16}$$

$$\frac{d}{d\lambda}v^2 = \frac{d}{d\lambda}vv = vv' + v'v = 2v \cdot v' \tag{17}$$

$$\frac{d}{d\lambda}|v| = \frac{d}{d\lambda}\sqrt{v^2} = \frac{v \cdot v'}{|v|} \tag{18}$$

therefore

$$\begin{aligned} \frac{d}{d\lambda}\frac{v}{|v|} &= \frac{1}{2|v|}\left(v' - \frac{vv'v}{|v|^2}\right) \overset{(11)}{=} v'_{\perp v}/|v| \\ &\equiv \frac{(v' \wedge v)\,v^{-1}}{|v|} \equiv \frac{(v' \wedge v)\,v}{|v|^3} \end{aligned} \tag{19}$$

i.e. the derivative of a unit vector field v along a curve is perpendicular to the original field. A visualization of this behavior is shown later in Fig. 5. The same fact is evident from $v \cdot \frac{d}{d\lambda}\frac{v}{|v|} = 0$, noticing eqn. (18) becomes zero for a unit vector $|v| = 1$,

Consecutively applying the operations of derivation and normalization on the tangential vectors of a curve leads to a systematic scheme allowing to study a curve's properties, known as the Frenet-Serret formulas.

2.3 Frenet-Serret Formulae

Curvature of a Curve The curvature κ of a curve is defined as the magnitude of the rate of change of the unit tangent vector $t$with respect to arc length:

$$\kappa := |\dot{t}| = \left|\frac{d}{ds}\,t\right| \equiv \frac{1}{|q'|}\,|t'| \tag{20}$$

The derivative of the tangent vector is perpendicular to q' by means of (19):

$$t' = \frac{q''_{\perp q'}}{|q'|} \equiv \frac{(q'' \wedge q')\,q'}{|q'|^3} \quad . \tag{21}$$

The curvature can thus be seen as the rejection (perpendicular component) of the second derivative $q'' = d/ds\, q'$ by the velocity q' normalized by the speed:

$$\kappa = \left|\frac{q''_{\perp q'}}{|q'|^2}\right| = \frac{|(q'' \wedge q')\,q'|}{|q'|^4} \tag{22}$$

By construction the curvature κ is independent of the parameterization and is a measure that only depends on the line, as in (20) we differentiate with respect to arc length, not the curve parameter.

Relation to Vector Calculus By means of (14) we may express t' in (21), and thus $\dot{t}$, as a cross product,

$$\dot{t} = \frac{q''_{\perp q'}}{|q'|^2} = \frac{(q'' \wedge q')\,q'}{|q'|^4} = \frac{q' \times (q'' \times q')}{|q'|^4} \tag{23}$$

such that via $|a \times (b \times a)| = |a|\ |b \times a|$ we get the commonly shown formula for curvature as

$$\kappa = \frac{|q' \times (q'' \times q')\,|}{|q'|^4} = \frac{|q'|\ |q'' \times q'|}{|q'|^4} = \frac{|q'' \times q'|}{|q'|^3}\ . \tag{24}$$

Normal Vector and Osculating Bivector Derivation and normalization of the tangential vector $t = q'/|q'|$ yields the *normal* unit vector, a quantity independent of the curve parameterization:

$$n := \frac{t'}{|t'|} \equiv \frac{\dot{t}}{|\dot{t}|} \overset{(20)}{\equiv} \frac{1}{\kappa}\dot{t} \overset{(6)}{=} \frac{1}{\kappa}\frac{1}{|q'|}t' \tag{25}$$

By definition of the curvature (20) we trivially arrive at the first Frenet-Serret equation:

$$\boxed{t' = |t'|\ n = |q'|\ \kappa\, n} \text{ or } \boxed{\dot{t} = \kappa\, n} \tag{26}$$

The tangent and normal vector define the *osculating plane* of the curve, called the binormal vector $t \times n$ in vector calculus. It corresponds to a bivector in Geometric Algebra ($t \cdot n = 0$):

$$b := tn = t \wedge n = -n \wedge t = -nt \quad . \tag{27}$$

This "osculating bivector" b is a unit bivector fulfilling $b^2 = -1$. The associated "curvature bivector" $\kappa b = t \wedge \dot{t}$ fulfills $(t \wedge \dot{t})^2 = -\kappa^2$.

Relation to Vector Calculus The normal vector expressed in derivatives of the curve q becomes in Euclidean vector calculus, using (23) and (24):

$$n = \frac{q' \times (q'' \times q')}{|q'|\ |q'' \times q'|} \tag{28}$$

Torsion Trivector The change of normal vector yields the form of a unit vector derivative (19):

$$n' = \frac{t''_{\perp t'}}{|t'|} \equiv \frac{(t'' \wedge t')\,t'^{-1}}{|t'|} \tag{29}$$

To compute the change of the osculating bivector, we utilize the Leibniz rule on (27), notice that t' and n are parallel by eqn. (26) and reorder terms to find

$$\begin{aligned}\dot{b} &= \frac{1}{|q'|}\ \big(\underbrace{t' \wedge n}_{0} + t \wedge n'\big) = \frac{1}{|q'|}\ t \wedge \frac{(t'' \wedge t')\,t'^{-1}}{|t'|} \\ &= \underbrace{-\frac{1}{|q'|}\frac{t \wedge t' \wedge t''}{|t'|^2}}_{=:\tau}\ \underbrace{t'/|t'|}_{n} = -\frac{t \wedge \dot{t} \wedge \ddot{t}}{\kappa^2}\ n =: -\tau n\end{aligned} \tag{30}$$

This is the third Frenet-Serret equation,

$$\boxed{b' = -\tau\ n\ |q'|} \ \ or\ \ \boxed{\dot{b} = -\tau\ n} \tag{31}$$

which in this formulation relates the change of the osculating bivector to the normal vector via the torsion trivector τ. With the geometric product being invertible we can easily express n by means of b by noting $t^{-1} = t$ due to $|t| = 1$, finding $n = t^{-1}b = tb$. Derivation yields

$$\dot{n} = \dot{t}b + t\dot{b} = \kappa n\, b - t\ \tau n \tag{32}$$

and using $nb = ntn = -nnt = -t$ with $t\tau = \tau\, t$ provides via $tn = b$ the second Frenet-Serret equation:

$$\boxed{\dot{n} = -\kappa\, t - \tau\, b} \tag{33}$$

Here b is a bivector (describing curvature) and τ is a trivector (describing torsion). It is evident that τ is not a scalar, but a pseudo-scalar - it changes sign under reflection: a helix with positive torsion seen in a mirror exhibits negative torsion.

Relation to Vector Calculus In a chart the torsion trivector will be expressed as a three-indexed object $\tau = \tau_{ijk}\ \partial_i \wedge \partial_j \wedge \partial_k$. These are 2^n components, but in three dimensional Euclidean space they reduce to a single number and the torsion trivector can be associated with a scalar $|\tau|$ by means of the hodge-star operator as $\star\tau = |\tau|\Omega$. It expresses the torsion trivector relative to an orientation Ω describing the left-handedness or right-handedness of the chosen coordinate system. We can express eqn. (33) through the vector dual $\vec{b}$ to the bivector $b = \star\vec{b} = \Omega\vec{b}$, which due to $\Omega^2 = -1$ yields the usual Frenet-Serret equation for vectors:

$$\dot{n} = -\kappa\, t - |\tau|\Omega\ \ \Omega\vec{b} = -\kappa\, t + |\tau|\ \vec{b} \quad . \tag{34}$$

3 VISUALIZING CURVES

3.1 Integral Curves

Given a vector field $v : M \to T(M) : q \mapsto v(q)$ a curve $q(\lambda)$ is an integral curve on this vector field if it fulfills $\frac{d}{d\lambda}q = v\big(q(\lambda)\big)$. The properties of the integral curve are determined by the vector field itself. Curvature and torsion can be computed directly as curvature and torsion *fields* from the vector field [11] based on the vector field's Jacobian. As the previous has shown, these are actually bivector and trivector fields, not scalar fields.

3.2 Verification Vector Field

Verification of computational methods on behalf of analytical test data sets is of utmost importance. Here, a vector field is used for verifying the computational methods and is constructed to yield clearly defined results (stream lines) in the form of circles: $v = \partial_\varphi = (-y, x, 0)/\sqrt{x^2 + y^2}$. To check the independence of the curve parameterization we provide a non-constant velocity depending on the angle relative to the coordinate system: $v = \left[1 + A\sin\left(\varphi \arctan\frac{y}{x}\right)\right]\partial_\varphi$ with A an amplitude of the modification and φ a modification factor for the angular dependency. With $A = 0.9$ and $\varphi = 1.0$ we get a vector field that is nearly zero for $y < 0$ and is large up to $|v| = 1.9$ for $y > 0$, as show in Fig. 1. The integral lines of this vector field are closed circles. To simulate data stemming from a numerical simulation the vector field is sampled to uniformly spaced locations and interpolated to arbitrary locations where the integral line passes through.

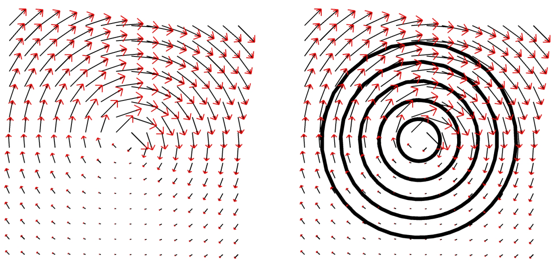


Figure 1: 2D slice of an axial vector field with non-constant (but non-vanishing) velocity sampled on a grid of 16^3 points (left image). The integral lines of this vector field are closed circles (right image) of constant curvature, with curvature increasing toward the center.

3.3 Fields for Visual Analysis

The differential geometric treatment of curves systematically leads to a set of fields that allow to study the properties of a curve. Some of these fields are local quantities, i.e. they can be computed from the properties of a curve at each point and its neighborhood, but are otherwise independent of global quantities which depend on the entire shape of the curve. Both local and global quantities are of interest. Some of them are dependent on the parameterization and others are pure line quantities which do no change under re-parameterization.

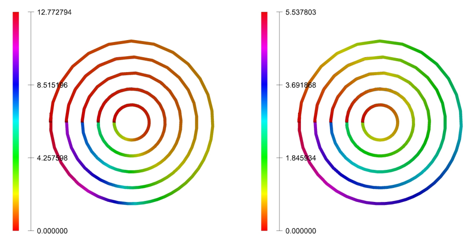
1. Proper time: $T = \int 1/|\dot{q}(\lambda)|d\lambda$
2. Arc length: $s = \int ds$
3. Velocity: q'
4. Coordinate Acceleration: q''
5. Energy: $E = |q'|^2/2$
6. Tangential vector: $t = q'/|q'|$
7. Normal vector: $n = t'/|t'|$
8. Osculating bivector: $b = tn$
9. Curvature: $\kappa = |\dot{t}| = |(q'' \wedge q')\, q'|/|q'|^4$
10. Curvature bivector $\kappa b = t \wedge \dot{t}$
11. Torsion trivector: $\tau = (t \wedge \dot{t} \wedge \ddot{t})/\kappa^2$
12. Torsion: $|\tau| = |\dot{b}| = |t \wedge \dot{t} \wedge \ddot{t}|/\kappa^2$

These quantities will be of type scalar, vector, bivector and trivector, each of these types requiring a different kind of visualization method along the curve. Scalar fields are commonly displayed via color-coding, vector fields via arrow icons. With bivectors and trivectors being elements of Geometric Algebra beyond the usual vector calculus, there are no common visualization methods for these types of fields. However, Frenet Ribbons, discussed in 3.4, provide a direct visualization of the osculating bivector field.

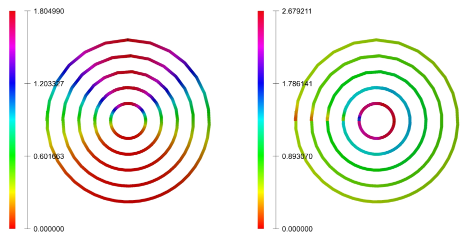
Scalar Fields The set of available scalar fields from the above set can be organized - for planar curves (zero torsion) - with respect to their properties of being local or global and their dependence on the curve parameterization (line quantity vs. curve quantity):

$$\begin{pmatrix} T & s \\ E & \kappa \end{pmatrix} \hat{=} \begin{pmatrix} global/curve & global/line \\ local/curve & local/line \end{pmatrix} \tag{35}$$

As demonstrated in Fig. 2, displaying these four quantities along a line provides four different views with complementary information. If we modify the input vector field by an arbitrary modulation of its amplitude, then the right column of Fig. 2 will remain unchanged, while only the left column will undergo changes. On the other hand, the lower row will be independent of the placement of integral seed points. Mapping proper



(a) Proper Time- T (b) Arc Length - s

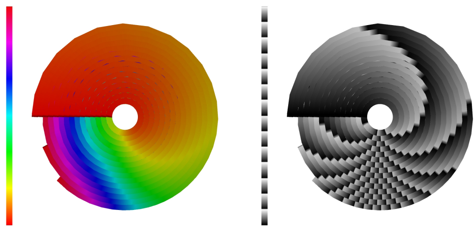


(c) Energy - E (d) Curvature - κ

Figure 2: Visualization matrix of scalar fields on a curve: proper time T, arc length s, energy E and curvature κ. Upper row are global (integral) quantities, right column are independent of parameterization.

time to colors provides a notion of the time that a particle requires to reach a certain point on this trajectory.

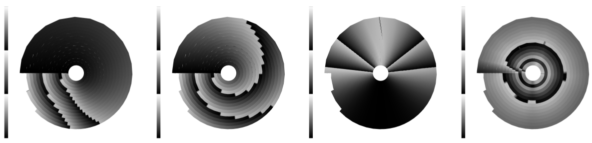
A colormap that uses gradient steps is furthermore able to emphasize the increments of proper time along the line, even in mere grayscale depiction. It provides a visual indication of the velocity and therefore the original vector field. Particles are traveling slower in the lower



(a) Value $\rightarrow$ color hue (b) Value $\rightarrow$ intensity steps

Figure 3: Visualizing a dense set of curves: Proper time with color map, showing advancement of time along the curves, and proper time with "zebra" colormap, depicting the velocity along the curve.

section of Fig. 3, which is conveyed better by the chosen gradient colorization. Fig. 4 shows the scalar fields with the "zebra" map applied.



(a) T (b) s (c) E (d) κ

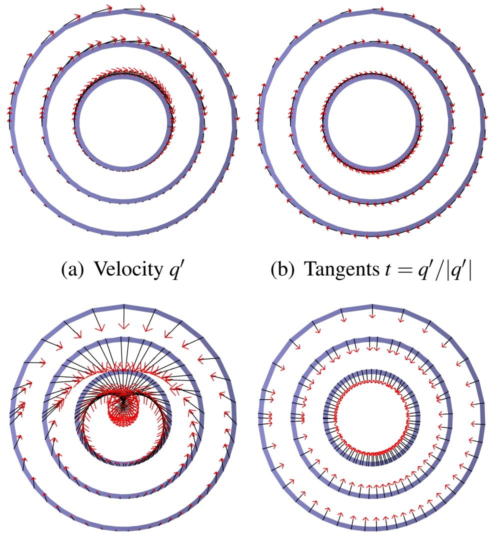
Figure 4: "Zebra" colorization scheme applied to the matrix of scalar fields (Fig. 2) for a dense set of curves: proper time, arc length, energy, curvature.

Vector Fields The velocity q' along a curve is supposed to be identical with the value of a vector field $v(q(s))$ if q is an integral curve. For vector fields given on discrete points its visualization may provide insight into the behavior of the interpolation algorithm, as discussed in 3.5, which in particular is non-trivial for curvilinear grids [9] [4] [10].

Same as with scalar fields, we can distinguish among curve and line quantities based on the dependency of a vector field on the curve parameterization. Since all vector fields "live" in the tangential space $T_p(M)$, they are local by nature. The notion of global vs. local is hereby replaced by order of derivation, firstly considering first and second order:

$$\begin{pmatrix} q' & t \\ q'' & n \end{pmatrix} \triangleq \begin{pmatrix} 1^{st}/curve & 1^{st}/line \\ 2^{nd}/curve & 2^{nd}/line \end{pmatrix} \tag{36}$$

The corresponding vector fields are shown in Fig. 5. Note that the acceleration q'', Fig. 5(c) is not normal to the velocity q', Fig. 5(a), but lays in the same plane as the normal vectors n, Fig. 5(d). We can thus see Fig. 5 as a direct visualization of eqn. (21) which states that the direction of the normal vector is given by the projection of the acceleration on the velocity $n \propto q''{}_{\perp q'}$.



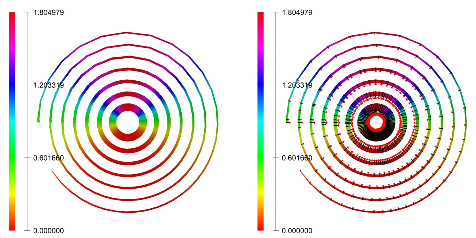
(a) Velocity q' (b) Tangents $t = q'/|q'|$

(c) Acceleration q'' (d) Normals $n = t'/|t'|$

Figure 5: Visualization matrix of vector fields on a curve: velocity and acceleration (left column) depend on the curve's parameterization, tangents and normals (right column) are pure geometrical quantities. Upper row includes first order derivatives of the curve, lower row is based on second order derivations.

3.4 Visualizing the Curvature Bivector: Frenet Ribbons

A Frenet Ribbonis a direct visualization (Fig. 6) of the curvature bivector field $\kappa\ b = t \wedge \dot{t}$ along a curve q. The Frenet Ribbon is the surface generated by sweeping the tangential derivative vector t' along the curve q. Its width depicts the curvature $\kappa = |\dot{t}|$, the location of the surface relative to the curve q depicts the sign of the curvature, since osculating plane is described by the bivector $t \wedge \dot{t} = -\dot{t} \wedge t$.



(a) Frenet Ribbon (b) Ribbon and normals

Figure 6: A Frenet Ribbon is generated by sweeping the curve normal vectors along the curve. Colorization by energy.

Using modern graphics hardware, Frenet Ribbons are very suitable to be implemented using geometry shaders which allow generating the actual geometry completely on the GPU while just providing the line and normal vector information for each vertex. Consequently rendering Frenet Ribbons is about as fast as

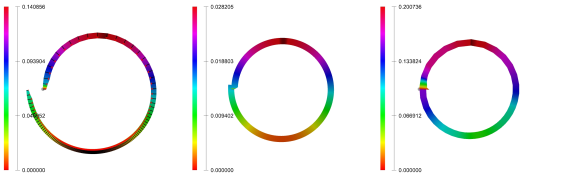


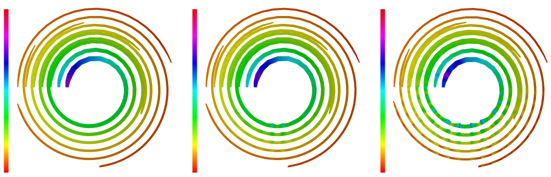
Figure 7: Accuracy and performance of integration methods: Euler integration with stepsize 1.0, 180 steps; stepsize 0.2, 720 steps; 8th order Runge-Kutta (DOP853), 40 steps.

drawing line primitives unless there occur huge polygons to be rendered due to locations of very high curvature. Usually rendering is possible in realtime with at least 30fps using a decent modern graphics card which supports geometry shaders.

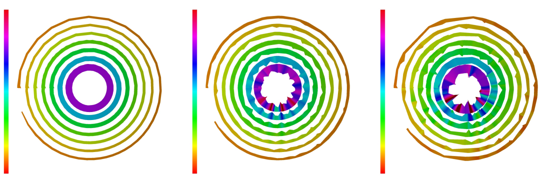
3.5 Numerics

Integration Method The forward Euler method is the most simple method to advance a point at a curve via $q(s+\Delta) = q(s) + \Delta\, v(q(s))$ for a constant step size Δ. It is known to always gives overshoots of the curve, which can be cured somewhat by reducing the stepsize Δ. But it is never able to achieve the same accuracy as an higher order integration schemes such as the DOP853 Runge-Kutta scheme of order 8^{th} with adaptive step size control [3], as demonstrated in Fig. 7. In theory, all line and curve quantities are supposed to be independent of the chosen method. In practice, they will differ.

Interpolation Scheme With vector field data given at an equidistant spatial sampling (“uniform grid”) it is required to interpolate grid points to a smooth location. With linear interpolation the discretized manifold is $\mathscr{C}^1$, first order derivatives become discontinuous which shows up visibly in the curvature (Fig. 8(c)). Cubic interpolation via Catmull-Rom splines yields a somewhat smoother curvature, Fig. 8(b), yet artifacts are still visible. While Euler integration yields inaccurate results, the DOP853 integrator exhibits a remarkable behavior when visualization curvature (Fig. 8): it apparently approaches the curve by a combination of “undershooting” and “overshooting”, which is more sensitive to the interpolation method.



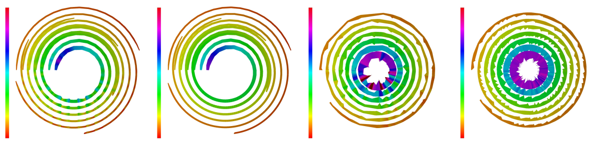
(a) analytic (b) bi/tri-cubic (c) bi/tri-linear



(d) analytic (e) bi/tri-cubic (f) bi/tri-linear

Figure 8: Curvature on Euler (upper row) and DOP853 integration (lower row).

Discretization Resolution The sampling density of an analytic function influences the accuracy of an integration scheme. As depicted in Fig. 9, increasing the grid resolution does smooth out the curvature as computed by the Euler scheme. For the more accurate DOP853 scheme however the “meandering” behavior as observed in Fig. 8 remains, just on a smaller scale.



(a) Euler, 16^3 (b) Euler, 64^3 (c) DOP, 16^3 (d) DOP, 64^3

Figure 9: Dependency of Frenet Ribbons and curvature on grid resolution.

Differentiation Scheme We implemented curvature computation directly by means of the definition (20) $\kappa = |\dot{t}|$. Given a discrete set of N vertices along a line with $p_i \in M$, $v_i \in T_{p_i}(M)$ for $i \in [0, N)$, we arrive at

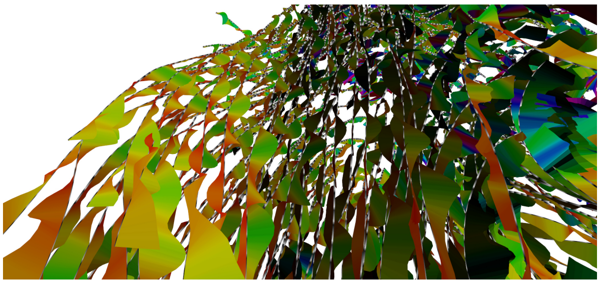
$$
\begin{aligned}
t_- &:= v_{i-1}/|v_{i-1}| & \Delta_- &:= |p_i - p_{i-1}| \\
t_0 &:= v_i/|v_i| & & \\
t_+ &:= v_{i+1}/|v_{i+1}| & \Delta_+ &:= |p_{i+1} - p_i| \\
\dot{t}_- &:= \frac{t_0 - t_-}{\Delta_-} & \dot{t}_+ &:= \frac{t_+ - t_0}{\Delta_+} \\
\dot{t} &= \frac{1}{2}(\dot{t}_- + \dot{t}_+) & \Rightarrow \quad \kappa &= |\dot{t}|
\end{aligned}
$$

This is a second order scheme for differentiation, where division by the arc length Δ of a finite curve segment yields $\dot{t}$ (not t'), taking into account non-equidistant step sizes between succeeding points on the line. In the special case of constant step size, such as with Euler integration, the term t_0 will be discarded. When using the adaptive DOP853 scheme this symmetric formula using the tangential vector at $i-1$ and $i+1$ is important to yield smooth results. Boundary conditions ($i = 0$ and $i = N-1$) need to be treated differently. The computation of the torsion trivector is a direct implementation of eqn. (30), requiring one more numerical differentiation of t':

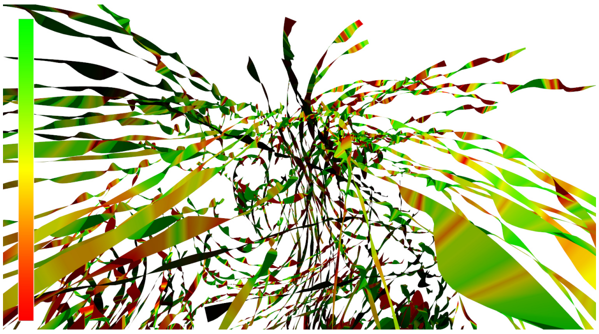
$$
\ddot{t} := \frac{\dot{t}_+ - \dot{t}_-}{\Delta_+ + \Delta_-} \quad \Rightarrow \quad \tau = \frac{t_0 \wedge \dot{t} \wedge \ddot{t}}{\kappa^2} \tag{37}
$$

The method applied to a numerical dataset stemming from a large-scale computational fluid dynamic simulation is demonstrated in Fig. 10, showing Frenet Ribbons exposing the curvature bivector and torsion.

Curvature and torsion of pathlines are suitable indicators of the mixing quality of fluids [1]. Within a large numerical dataset their depiction via Frenet Ribbons is useful for data mining purposes as slight deviations in curvature and torsion show up prominently.



(a) Frenet Ribbons with Color-Encoded Curvature



(b) Frenet Ribbons with Color-Encoded Torsion

Figure 10: Frenet Ribbons in a numerical vector field. Ribbons color-encoded by curvature or torsion, lines by proper time with gradient map.

4 CONCLUSION

In this article we have reviewed the Frenet-Serret equations describing the Differential Geometry of curves in the formalism of Geometric Algebra. This leads to a more intuitive formulation of curvature as a bivector and torsion as a trivector, explaining sign changes under reflection. The formalism is valid also in higher dimensional spaces, thereby generalizing vector algebra employing cross-products and quaternion formulations. The apparatus is applied to the numerical computation of integral curves in discretized vector fields and investigated for its dependency on numerical artifact such as interpolation scheme, integration method, sampling resolution and differentiation scheme. A real-world example is demonstrated on behalf of a dataset from computational fluid dynamics where Frenet Ribbons visualize the trajectories of test particles, exhibiting curvature and torsion.

5 ACKNOWLEDGMENTS

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