Operators for Multi-Resolution Morse Complexes in Arbitrary Dimensions

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Abstract

Ascending and descending Morse complexes, defined by the critical points and integral lines of a scalar field f defined on a manifold M, induce a subdivision of M into regions of uniform gradient flow, and thus provide a compact description of the morphology of f on M. We propose a dual representation for the ascending and descending Morse complexes of f in arbitrary dimensions in terms of an incidence graph. We describe atomic simplification and refinement operators on the Morse complexes and we investigate the effect of those operators on the graph-based representation of the two complexes. Simplification and refinement operators form a basis for a hierarchical multi-resolution representation of Morse complexes, from which it will be possible to dynamically extract representations of the morphology of the scalar field f over M, at both uniform and variable resolutions.

1 Introduction

Representing morphological information extracted from discrete scalar fields is a relevant issue in several application domains, including terrain modeling, volume data analysis and visualization, and time-varying 3D scalar fields. Morse theory offers a natural and intuitive way of analyzing the structure of a scalar field f as well as of compactly representing the scalar field through a decomposition of the domain of f into meaningful regions associated with the critical points of the field. The ascending and the descending Morse complexes are defined by considering the integral lines emanating from, or converging to the critical points of f, while the Morse-Smale complex describes the subdivision of M into parts characterized by a uniform flow of the gradient between two critical points of f.

Structural problems in Morse and Morse-Smale complexes, like over-segmentation in the presence of noise, or efficiency issues arising because of the very large size of the input data sets, can be faced and solved by defining simplification operators on such complexes and on their morphological representations.

Here, we present atomic operators for simplifying and refining Morse complexes. Such operators are defined in arbitrary dimensions and affect a constant number of entities in the Morse complexes. We show in [6] that the simplification operators together with their refinement ones define a basis for simplifying Morse (and Morse-Smale) complexes. Moreover, the general cancellation operator defined in Morse theory [19] can be expressed as a suitable combination of our operators.

We represent the ascending and descending Morse complexes as an incidence graph. This representation is based on encoding the incidence relations of the cells of the Morse complexes, and exploits the duality between the ascending and descending complexes. We define the effect of the simplification and of the refinement operators on the incidence-based dual representation of the descending and ascending Morse complexes. The two operators are defined in a dimension independent way, and their effect on the graph-based representation of the Morse complexes is easy to describe and implement. Moreover, they form the basis for the definition of a hierarchical model of the Morse complexes. A hierarchical representation of the morphology of a scalar field is critical for interactive analysis and exploration in order to maintain and analyze characteristic features at different levels of abstraction.

The remainder of the paper is organized as follows. In Section 2, we review some basic notions on Morse theory and Morse complexes. In Section 3, we discuss some related work. In Section 4, we describe a dual incidence-based representation of the Morse complexes. In Sections 5 and 6, we present simplification and refinement operators respectively and we describe their effect on the incidence-based representation of the Morse complexes. Finally, in Section 7, we draw some concluding remarks and discuss current and future work.

2 Morse Theory and Morse Complexes

Morse theory studies the relationship between the topology of a manifold M and the critical points of a scalar (real-valued) function defined on the manifold (for more details on Morse theory, see [19, 20]).

Let f be a C^2 real-valued function defined over a closed compact n-manifold M. A point p is a *critical point* of f if and only if the gradient $\nabla f = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$ (in some local coordinate system around p) of f vanishes at p. Function f is a *Morse function* if all its critical points are non-degenerate (i.e. the Hessian matrix $Hess_p f$ of the second derivatives of f at p is non-singular). The number i of negative eigenvalues of $Hess_p f$ is called the *index* of critical point p, and p is called an *i-saddle*. A 0-saddle, or an n-saddle, is also called a *minimum*, or a *maximum*, respectively. An *integral line* of f is a maximal path which is everywhere tangent to the gradient of f. Each integral line connects two critical points of f, called its *origin* and its *destination*.

Integral lines that converge to (originate at) a critical point p of index i form an i-cell ($(n-i)$-cell) p called a *descending (ascending) cell*, or *manifold*, of p. The descending and ascending cells decompose M into *descending* and *ascending Morse complexes*, denoted as Γ_d and Γ_a, respectively, see Figure 1 (a) and (b) for a 2D example. We will denote as p the descending i-cell of an i-saddle p. A Morse function f is called a *Morse-Smale function* if the descending and the ascending manifolds intersect transversally. A *Morse-Smale complex* is defined by the connected components of the intersection of descending and ascending Morse complexes. If f is a Morse-Smale function, then complexes Γ_a and Γ_d are dual to each other.

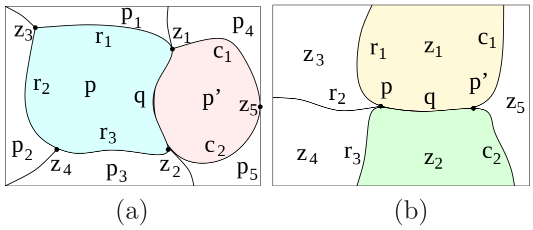


Figure 1: A portion of a descending Morse complex in 2D, with the descending cells of maxima p and p' highlighted (a), and the dual ascending Morse complex, with the ascending cells of minima z_1 and z_2 highlighted (b).

3 Related Work

In this Section, we review related work on morphological representations of scalar fields provided by Morse or Morse-Smale complexes. We concentrate on two topics, which are relevant to the work presented here, namely: computation and simplification of Morse and Morse-Smale complexes.

Several algorithms have been proposed in the literature for decomposing the domain of a 2D scalar field f into an approximation of a Morse, or a Morse-Smale, complex. Recently, some algorithms in higher dimensions have been proposed. For a review of the work in this area, see [3].

The extraction of critical points of a scalar field f defined on a simplicial mesh has been investigated in 2D [2, 21], and in 3D [15, 25, 26, 24, 12] as a basis for computing Morse and Morse-Smale complexes. Algorithms for decomposing the domain M of f into an approximation of a Morse, or of a Morse-Smale complex in 2D can be classified as *boundary-based* [1, 4, 13, 22, 23], or *region-based* [5, 9]. In [12], an algorithm for extracting the Morse-Smale complex from a tetrahedral mesh is proposed. The algorithm, while interesting from a theoretical point of view, exhibits a large computation overhead, as discussed in [18].

Discrete methods rooted in the discrete Morse theory proposed by Forman [14] are computationally more efficient. In [9], a dimension-independent approach based on region growing has been proposed which implements the discrete gradient approach and computes the descending and the ascending Morse complexes. In [18], a region growing method, inspired by the watershed approach, has been proposed to compute the Morse-Smale complex. In [16], a Forman gradient vector field V is defined, and an approximation of the Morse-Smale complex is computed by tracing the integral lines defined by V.

One of the major issues that arise when computing a representation of a scalar field as a

Morse, or as a Morse-Smale, complex is the over-segmentation due to the presence of noise in the data sets. *Simplification algorithms* have been developed in order to eliminate less significant features from a Morse-Smale complex. Simplification is achieved by applying an operator called *cancellation*, defined in Morse theory [19]. It cancels pairs of critical points of f, in the order usually determined by the notion of *persistence*, which is the absolute difference in function values between the paired critical points [13]. In 2D Morse-Smale complexes, the cancellation operator has been investigated in [4, 13, 17, 23, 27]. The cancellation operator on Morse-Smale and Morse complexes of a 3D scalar field has been investigated in [17] and [7], respectively.

4 A Dual Incidence-Based Representation for Morse Complexes

In this Section, we discuss a dual representation for the ascending and the descending Morse complexes Γ_a and Γ_d, that we call the *incidence-based representation*. The underlying idea is that we can represent both the ascending and the descending complex as a graph by considering the boundary and co-boundary relations of the cells in the two complexes. In the discrete case, we consider a representation for the simplicial mesh Σ which generalizes an indexed data structure commonly used for triangle and tetrahedral meshes [10], and we relate the two representations into the incidence-based data structure.

Recall that there is a one-to-one correspondence between i-saddles p and i-cells p in the descending complex Γ_d, and dual $(n-i)$-cells in the ascending complex Γ_a, $0 \leq i \leq n$. We exploit this duality to define a representation which encodes both the ascending and the descending complexes at the same time as an *incidence graph* [11]. The incidence graph encodes the cells of a complex as nodes, and a subset of the boundary and co-boundary relations between cells as arcs. The *incidence graph* associated with an n-dimensional descending Morse complex Γ_d (and with an ascending Morse complex Γ_a) is a graph $G = (N, A)$, in which

1. the set of nodes N is partitioned into $n+1$ subsets N_0, N_1,...,N_n, such that there is a one-to-one correspondence between nodes in N_i (which we will call *i-nodes*) and the i-cells of Γ_d (and thus the $(n-i)$-cells of Γ_a),

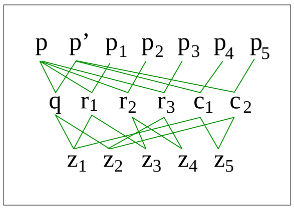


Figure 2: A portion of the incidence graph encoding the connectivity of descending and ascending Morse complexes illustrated in Figure 1 (a) and (b), respectively

2. there is an arc joining an i-node p with an $(i+1)$-node q if and only if the corresponding cells p and q differ in dimension by one, and p is on the boundary of q in Γ_d (q is on the boundary of p in Γ_a),

3. each arc connecting an i-node p to an $(i+1)$-node q is labeled by the number of times i-cell p (corresponding to i-node p) in Γ_d is incident to $(i+1)$-cell q (corresponding to $(i+1)$-node q) in Γ_d.

Attributes are attached to the nodes of the incidence graph, containing information about geometry, and function values, while arcs have no associated (geometric) attributes. Note that the incidence graph provides also a combinatorial representation of the 1-skeleton of a Morse-Smale complex. Figure 2 shows a portion of the incidence graph encoding the connectivity of the descending Morse complex in Figure 1 (a), and of the ascending Morse complex in Figure 1 (b).

We have designed and implemented a data structure based on the incidence graph by encoding this latter as a standard adjacency list. We associate with each 0-node p (corresponding to a minimum) a list of the n-simplexes of a simplicial complex Σ forming the corresponding ascending n-cell of p. Dually, we associate with each n-node p (corresponding to a maximum) a list of the n-simplexes forming the corresponding descending n-cell of p. The resulting data structure is the *incidence-based representation*.

5 Simplification Operators

In Morse theory, a general cancellation operator has been defined that allows eliminating any pair of critical points of consecutive index which are connected by a unique integral line [19]. One of the drawbacks of such operator, when applied to a Morse-Smale complex, is that the number of cells in the complex can increase, and, when applied to a Morse complex, the number of incidences among cells can also increase.

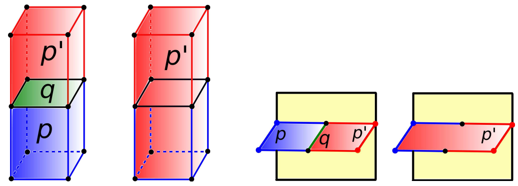
In [8], we have defined two dual simplification operators in arbitrary dimensions, which we call *removal* and *contraction*. The two simplification operators are defined in a dimension-independent way. They are defined by imposing constraints on a cancellation operator, that allow us to avoid creating new cells in the Morse-Smale complex or new incidences in the Morse ones. The two operators form a complete set of basic operators for simplifying Morse complexes on a manifold M, as detailed in [6]. Moreover, the classical cancellation operator [19] can be seen as a macro-operator and expressed as a sequence of our atomic operators. A persistence value is associated with a simplification operator, and thus we apply simplifications in order of increasing persistence [13].

The first operator, called a *removal* of index i, $1 \leq i \leq n-1$, removes an i-saddle q and an $(i+1)$-saddle p, provided that q is connected by a unique integral line to an $(i+1)$-saddle p, and to exactly one other $(i+1)$-saddle p' different from p, or to just one $(i+1)$-saddle p. In the first case, a removal of q and p is denoted as $rem(p, q, p')$, while in the second case as $rem(p, q, \emptyset)$. The second operator, that we call a *contraction* of index i, $1 \leq i \leq n-1$, removes an i-saddle q and an $(i-1)$-saddle p provided that q is connected by a unique integral line to an $(i-1)$-saddle p, and to exactly one other $(i-1)$-saddle p' different from p, or to just one $(i-1)$-saddle p. In the first case, a contraction of q and p is denoted as $con(p, q, p')$, and in the second case as $con(p, q, \emptyset)$. For the sake of simplicity, we discuss here only removals and contractions of the first kind.

5.1 Simplification on Morse complexes

The removal and contraction operators have a dual effect on the descending and the ascending Morse complexes. The effect of a contraction of index i on Γ_d (Γ_a) is the same as the effect of a removal of index $n-i$ on Γ_a (Γ_d). For the sake of brevity, we describe the effect of the two operators on the descending Morse complex only.

The effect of a removal $rem(p, q, p')$ on the descending Morse complex Γ_d is as follows: i-cell q, corresponding to i-saddle q is deleted and $(i+1)$-cell p, corresponding to $(i+1)$-saddle p is merged into $(i+1)$-cell p', which corresponds to $(i+1)$-saddle p'. A contraction $con(p, q, p')$ deletes i-cell q and merges $(i-1)$-cell p into $(i-1)$-cell p' in Γ_d. i-cell q is contracted, and each i-cell in the co-boundary of p is extended to include a copy of



(a) (b)

Figure 3: Portion of a 3D descending Morse complex before and after a removal $rem(p, q, p')$ of index 2 (a), and of index 1 (b).

i-cell q, i.e., each i-cell in the co-boundary of p is, after contraction, the union of itself with i-cell q.

The 2D case is simple, as our operators reduce to a minimum-saddle or a maximum-saddle cancellation operator. In 2D, there are exactly one removal and exactly one contraction operator (both of index 1). A removal deletes a 1-cell (saddle) q, and merges the two 2-cells (maxima) which shared q. It is the same as a maximum-saddle cancellation. A contraction contracts a 1-cell (saddle) q and collapses the two 0-cells (minima) which bounded q. It corresponds to a minimum-saddle cancellation. Note that both operators involve an extremum and a saddle.

In 3D, there are two removal and two contraction operators. A removal of index 2 involves a 2-saddle q and a maximum p (it is a maximum-2-saddle cancellation). In the descending complex, it removes a 2-cell q, and merges 3-cell p into a unique 3-cell p' incident in q and different from p, as illustrated in Figure 3 (a). A removal of index 1 involves a 1-saddle q and a 2-saddle p. It is defined only if 1-cell q is incident to exactly two different 2-cells p and p'. It removes 1-cell q and merges 2-cell p into 2-cell p', as illustrated in Figure 3 (b). Thus, it is a special case of a 1-saddle-2-saddle cancellation.

5.2 Simplification on the Incidence Graph

A removal, or contraction, simplification on the Morse complexes induces a modification on the incidence graph $G = (N, A)$ representing such complexes, that we call a *simplification modification*. Each simplification modification can be expressed as a deletion of two nodes p and q from N, and a replacement of a subset A^+ of the arcs in A with another subset A^- of arcs. For the sake of brevity, we will consider only a removal $rem(p, q, p')$ of index i, $1 \leq i \leq n-1$.

Let $G = (N, A)$ be the incidence graph representing both the descending and the ascending Morse complexes Γ_d and Γ_a before a removal

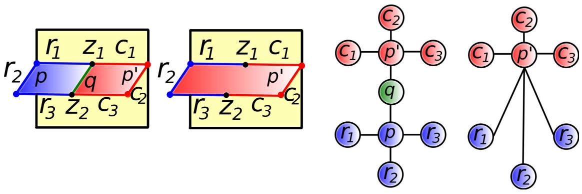


Figure 4: Removal $rem(p, q, p')$ on a 3D descending Morse complex (a) and on the corresponding incidence graph (b).

$rem(p, q, p')$. Then,

- i-node q is connected through an arc in A to exactly two different $(i+1)$-nodes p and p', such that the label of arcs (q, p) and (q, p') is 1, and to an arbitrary number of $(i-1)$-nodes from a set $Z = \{z_h, h = 1, .., h_{max}\}$;
- node p is connected to an arbitrary number of i-nodes from a set $R = \{r_j, j = 1, .., j_{max} : r_j \neq q\}$, and to an arbitrary number of $(i+2)-$nodes from a set $S = \{s_k, k = 1, .., k_{max}\}$;
- node p' is connected to an arbitrary number of i-nodes from a set $C = \{c_l, l = 1, .., l_{max} : c_l \neq q\}$, and to an arbitrary number of $(i+2)$-nodes from a set $D = \{d_m, m = 1, .., m_{max}\}$.

These conditions translate the feasibility condition of a removal operator.

For example, before the removal $rem(p, q, p')$, illustrated in Figure 4 (b), 1-node q is connected to exactly two different 2-nodes p and p' (corresponding to 2-saddles), and to two 0-nodes z_1, and z_2 (corresponding to minima), which are not shown in the Figure. 2-node p is connected to 1-nodes r_1, r_2 and r_3 and 2-node p' is connected to 1-nodes c_1, c_2 and c_3. Nodes p and p' are connected to exactly the same 3-nodes s_1 and s_2, which are not shown in the Figure.

As an effect of a removal $rem(p, q, p')$ on G, nodes p and q are deleted, as well as all the arcs incident into q, and all the arcs incident into p and connecting p to $(i+2)$-nodes in S. All the arcs incident into p and connecting p to i-nodes in R (with the exception of arc (p, q)), become incident in p'. Note that the effect of a contraction on G is exactly the same as that of a removal, except for the fact that in a removal q is an i-node, and p and p' are $(i+1)$-nodes, while in a contraction q is an i-node and p and p' are $(i-1)$-nodes.

Thus, a simplification modification induced by a removal can be expressed as a local modification of the incidence graph $G = (N, A)$ which produces a graph $G' = (N', A')$, where:

- $N' = N \backslash \{p, q\}$ and
- $A' = A \backslash A^+ \cup A^-$, such that

- $A^+ = \{(q, p)\} \cup \{(q, p')\} \cup \{(q, z_h) : z_h \in Z\} \cup \{(p, r_j) : r_j \in R\} \cup \{(p, s_k) : s_k \in S\}$,
- $A^- = \{(p', r_j), r_j \in R\}$.

Nodes p, q, p', z_h, r_j and s_k are as described above. In addition, for each arc (p, r_j), $r_j \neq q$, such that (p', r_j) is also an arc in A (i.e., such that $r_j = c_l$, for some l), the label of arc (p', r_j) is increased by the label of arc (p, r_j). A contraction can be expressed as a modification of graph G in a completely dual fashion. Thus, a simplification modification on the incidence graph can be expressed as a pair (A^+, A^-), i.e, as the collections of the arcs which are removed (A^+) and which are inserted (A^-).

In the example in Figure 4, after the removal of 1-saddle q and 2-saddle p, nodes q and p are deleted from the incidence graph ($N' = N \backslash \{q, p\}$), arcs connecting q to p and p', and arcs connecting 1-node q to 0-nodes z_1 and z_2 (not illustrated in the Figure) are deleted, as are arcs connecting 2-node p to 3-nodes s_1 and s_2 (not illustrated in the Figure). Arcs connecting 2-node p to 1-nodes r_1, r_2 and r_3 are replaced by arcs connecting 2-node p' to 1-nodes r_1, r_2 and r_3.

The effect on the incidence-based representation, that is the incidence graph extended with the references to the underlying simplicial decomposition, is restricted to the incidence graph when a simplification does not involve an extremum. When we perform a removal $rem(p, q, p')$ of index $n-1$, then the set of n-simplexes forming the descending cell of p are merged into the set of n-simplexes forming the descending cell of p'. Dually, a contraction $con(p, q, p')$ of index 1 merges the n-simplexes of the ascending cell of p with n-simplexes of the ascending cell of p'.

6 Refinement Operators

We have defined two refinement operators [6], which are inverse of the two simplification operators discussed in Section 5. Thus, they have the effect of introducing an i-saddle and an $(i+1)$-saddle by splitting an existing i-saddle or an $(i+1)$-saddle. They are defined as an undo of the corresponding simplifications. Before performing a refinement, the situation around the two newly introduced saddles, i.e., around the corresponding cells in the Morse complexes, needs to be the same as it was at the time of the inverse simplification. Like the two simplification operators, the two refinement operators are dual to each other. The first operator, called an *insertion* of index i, splits an $(i+1)$-saddle p' into p' and an $(i+1)$-saddle p by inserting an i-saddle q. The second operator, called an *expansion* of index i, splits an $(i-1)$-saddle p' into p' and an $(i-1)$-saddle p by expanding an i-saddle q.

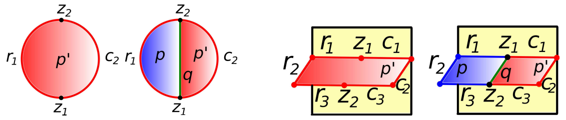
6.1 Refinement on Morse Complexes

In this Subsection, we discuss the effect of the refinement operators on the ascending and descending Morse complexes. In a descending complex Γ_d, an insertion of index i, denoted as $ins(p,q,p')$, which is the inverse (undo) of the removal $rem(p,q,p')$, consists of splitting an $(i+1)$-cell p' into two new $(i+1)$-cells p and p', by inserting an i-cell q into $(i+1)$-cell p'. i-cell q is shared by $(i+1)$-cells p and p'. For the correct application of the operator, we need to specify explicitly:

- the new cells p and q,
- the existing $(i+1)$-cell p',
- i-cells r_j in R, $j = 1,..,j_{max}$, which were on the boundary of $(i+1)$-cell p' before the insertion, and which are on the boundary of $(i+1)$-cell p after the insertion,
- $(i-1)$-cells z_h in Z, $h = 1,..,h_{max}$, which are on the boundary of i-cell q after the insertion, and
- $(i+2)$-cells s_k in S, $k = 1,..,k_{max}$, which are in the co-boundary of $(i+1)$-cell p after the insertion.

Note that the $(i+1)$-cells in the co-boundary of i-cell q after the insertion are exactly $(i+1)$-cells p and p'. The cells in Z (which will be on the boundary of i-cell q), in R (which will be on the boundary of $(i+1)$-cell p), and in S (which will be in the co-boundary of $(i+1)$-cell p) need to be the same as the corresponding cells on the boundary and in the co-boundary of p and q before the inverse removal $rem(p,q,p')$.

Figure 5 (a) shows an insertion $ins(p,q,p')$ of index 1 of 1-cell q and 2-cell p into 2-cell p' in a 2D descending Morse complex. It is specified by 1-cell q, 2-cells p and p', 0-cells z_1 and z_2 on the boundary of 1-cell q, and 1-cell r_1 on the boundary of 2-cell p. (The co-boundary of 2-cell p in 2D is empty.) Figure 5 (b) shows the effect of the insertion $ins(p,q,p')$ of index 1 in a 3D descending Morse complex. It is specified by 1-cell q, 2-cells p and p', 0-cells z_1 and z_2 on the boundary of 1-cell



(a) (b)

Figure 5: Insertion $ins(p,q,p')$ of index 1 on a descending Morse complex in 2D (a), and in 3D (b). It is specified by cells p, q and p', and cells in the immediate boundary and on the immediate co-boundary of the introduced cells p and q.

q, 1-cells r_1, r_2 and r_3 on the boundary of 2-cell p, and 3-cells s_1 and s_2 in the co-boundary of 2-cell p.

An expansion of index i, denoted as $exp(p,q,p')$, which is the inverse of contraction $con(p,q,p')$, consists of splitting an $(i-1)$-cell p' in Γ_d into two new $(i-1)$-cells p and p' by expanding a new i-cell q bounded by p and p'. It is specified by a list of cells on the immediate boundary and on the immediate co-boundary of the new cells p and q, which are the same as the corresponding cells before contraction $con(p,q,p')$.

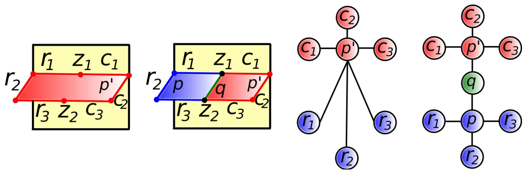
6.2 Refinement on the Incidence Graph

Like a simplification operator on Morse complexes, a refinement operator on Morse complexes induces a modification on the incidence graph $G' = (N', A')$ representing these complexes, that we call a *refinement modification*. Since a refinement operator is defined as an undo of the corresponding simplification operator, each refinement modification on the incidence graph is also an undo of the corresponding simplification modification. A refinement modification can be expressed as an insertion of two nodes p and q into N', and a replacement of a set A^- of arcs in A' with set A^+. Nodes p and q are the nodes which were eliminated by the inverse simplification modification, and A^- and A^+ are exactly the same sets of arcs which defined the inverse simplification modification. In other words, a refinement modification inverse to a simplification modification defined by (A^+, A^-) is defined by (A^-, A^+).

Specifically, given an insertion operator $ins(p,q,p')$, the corresponding refinement modification of the incidence graph $G' = (N', A')$ produces a graph $G = (N, A)$, where:

- $N = N' \cup \{p, q\}$ and
- $A = A' \backslash A^- \cup A^+$, such that

- $A^- = \{(p', r_j), r_j \in R\}$.



(a) (b)

Figure 6: (a) Insertion operator $ins(p,q,p')$ of a 1-cell q and 2-cell p in the 3D descending complex, and (b) the corresponding refinement modification of the incidence graph.

- $A^+ = \{(q,p)\} \cup \{(q,p')\} \cup \{(q,z_h) : z_h \in Z\} \cup \{(p,r_j) : r_j \in R\} \cup \{(p,s_k) : s_k \in S\}$.

Here, $(i-1)$-nodes $z_h \in Z$ correspond to $(i-1)$-cells on the boundary of i-cell q, i-nodes $r_j \in R$ correspond to i-cells on the boundary of $(i+1)$-cell p, and $(i+2)$-nodes $s_k \in S$ correspond to $(i+2)$-cells in the co-boundary of $(i+1)$-cell p. Arc (p',r_j) is removed from A if label of arc (p',r_j) minus label of arc (p,r_j) equals 0. Otherwise, arc (p',r_j) remains in A with label diminished by label of arc (p,r_j). An expansion can be expressed as a modification of graph G' in a completely dual fashion.

Figure 6 shows the effect of the refinement modification induced by an insertion $ins(p,q,p')$ of index 1 in 3D. Here, $Z = \{z_1, z_2\}$, $R = \{r_1, r_2, r_3\}$, and $S = \{s_1, s_2\}$. After the modification, the new 1-node q is connected to 2-node p', new 2-node p, and to 0-nodes z_1 and z_2 in Z. 2-node p is connected to 1-nodes r_1, r_2 and r_3 in R, and to 3-nodes s_1 and s_2 in S.

7 Concluding Remarks

We have presented a dimension-independent representation which encodes both the ascending and descending Morse complexes in a single combinatorial structure, the incidence-based representation. This is achieved by exploiting the duality of the two complexes which leads to an incidence graph representation of their connectivity. We have described simplification operators for generalizing Morse complexes in arbitrary dimensions and their inverse refinement operators. In particular, we have presented their effect on the incidence graph in a completely dimension-independent way. The simplification and refinement operators are the basic ingredients for the definition of a hierarchical representation for the dual Morse complexes in terms of the incidence graph, which will provide a description of the Morse complexes at different levels of abstraction.

Currently, we are working on a dimension-independent implementation of simplification and refinement operators on the incidence-based representation. Our next step is the design and implementation of a multi-resolution representation for the two Morse complexes by defining its encoding data structure, an algorithm for computing it based on iterative simplification, and a selective refinement algorithm for extracting adaptive Morse complexes.

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